Biclosed sets in combinatorics

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Thomas McConville

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor of Philosophy

Pavlo Pylyavskyy

August, 2015

Acknowledgements

I thank my advisor, Pasha Pylyavskyy, for his time, patience, and encouragement, and for presenting me with a wide variety of mathematical problems. I am indebted to Vic Reiner for generously sharing his knowledge and insight with me. His advice has been invaluable to me. I thank Nathan Reading for his encouragement and for his substantial influence on my work. I am grateful for the rest of the combinatorics faculty at the University of Minnesota, especially Gregg Musiker and Dennis Stanton. I thank Ravi Janardan for reviewing this thesis and his participation in my defense. I thank my coauthor, Al Garver. I am grateful for discussions with many others, especially Marcelo Aguiar, Cesar Ceballos, Tricia Hersh, Christophe Hohlweg, Jean-Philippe Labbé, Karola Mészáros, Vincent Pilaud, Hugh Thomas, and Nathan Williams. I would like to thank all of my colleagues at the University of Minnesota, particularly Alex Csar, Steven Collazos, Elise DelMas, Kevin Dilks, Theodosios Douvropolous, Emily Gunawan, Jia Huang, Jonas Karlsson, Shay Logan, Alex Miller, and Becky Patrias. I thank my girlfriend and my family for their love and support.

Dedication

To Ana and Kelly

Abstract

The *weak order* is the set of permutations of $[n]$ partially ordered by inclusion of *inversion sets*. This partial order arises naturally in various contexts, including enumerative combinatorics, hyperplane arrangements, Schubert calculus, cluster algebras, and many more. A fundamental result on the weak order is that the collection of maximal chains in any interval is connected by certain "local moves". Other notable features of the weak order are its lattice structure, its topology, and its geometry.

The collection of inversion sets of permutations is an example of a family of *biclosed sets*. This thesis focuses on extending various nice properties of the weak order to other posets of biclosed sets. Some of these collections of biclosed sets have appeared previously in the literature (e.g. [14], [19], [30], [48], [57], [58], [70], [88], [102]), while others seem to be new. We briefly summarize our main results below.

- (§3.1.3) We give a criterion on a closure operator which ensures that the poset of biclosed sets is a congruence-uniform lattice.
- (§4) The chambers of a real simplicial or supersolvable hyperplane arrangement are in natural bijection with biclosed subsets of hyperplanes.
- (§4, completing the proof in [81]) The graph of reduced galleries of a supersolvable hyperplane arrangement has diameter equal to the number of codimension 2 intersection subspaces.
- (§5) Chamber posets are semidistributive lattices if and only if they are crosscut-simplicial if and only if the arrangement is bineighborly.
- (§6) Every interval of the second Higher Bruhat order is either contractible or homotopy equivalent to a sphere.
- (§7) Every "facial" interval of the poset of reduced galleries of a supersolvable arrangement is homotopy equivalent to a sphere.
- (§8) The Grid-Tamari orders are congruence-uniform lattices.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

The set of permutations of $[n] := \{1, \dots, n\}$, denoted $\mathfrak{S}_n$, is a central object in algebraic combinatorics. The *inversion set* of a permutation of $[n]$ is the collection of pairs $\{i, j\},\ 1 \leq i < j \leq n$ such that j precedes i in the permutation. For example, the inversion set of 2314 is $\{\{1, 2\}, \{1, 3\}\}$. The *weak order* on $\mathfrak{S}_n$ is the set of permutations ordered by inclusion of inversion sets. The weak order may be defined similarly on any Coxeter group, replacing the set of pairs $\binom{[n]}{2}$ by the appropriate root system, which we review in §2.4. This poset may be realized as the skeleton of a polytope called the *permutahedron* whose faces are in bijection with the non-contractible intervals of the weak order.

If X is a subset of $\binom{[n]}{2}$, we say X is *closed* if $\{i, j\} \in X$ and $\{j, k\} \in X$ implies $\{i, k\} \in X$ for $1 \leq i < j < k \leq n$. We say X is *biclosed* (or *clopen*) if both X and $\binom{[n]}{2} - X$ are closed. Then a subset of $\binom{[n]}{2}$ is biclosed if and only if it is the inversion

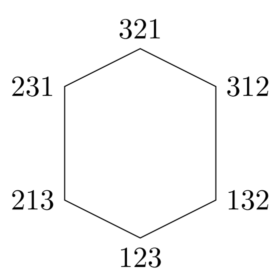


Figure 1.1: The weak order on permutations of $\{1, 2, 3\}$.

set of some permutation of $[n]$. In this terminology, the weak order is the set of biclosed subsets of the space $\binom{[n]}{2}$, ordered by inclusion.

More generally, if $\mathcal{T}$ is any finite closure space, we consider the set $\mathrm{Bic}(\mathcal{T})$ of *biclosed* subsets of $\mathcal{T}$, where X is biclosed if it is both closed and co-closed. The purpose of this work is to recast some of the nice properties of the weak order as properties of biclosed sets for other spaces. For the most part, our spaces are standard combinatorial objects such as finite root systems, d-element subsets of $[n]$, and paths in a graph. A broader study of the lattice structure of biclosed sets can be found in recent work of Santocanale and Wehrung [88].

Many interesting problems for biclosed sets and the weak order remain open for infinite root systems. Dyer's paper [30] presents a wide array of conjectures about biclosed sets in infinite root systems, which has been further studied in ([31], [48], [49]). We do not deal with any of these conjectures directly in this work, though we hope that a better understanding of the finite case will lead to some progress on these conjectures in the future.

1.1 Lattice structure

The weak order on $\mathfrak{S}_n$ was originally proved to be a lattice by Yanagimoto and Okamoto in [101]. This result was extended to finite Coxeter groups by Björner [10]. More recently, the weak order on a finite Coxeter group was shown to be a congruence-uniform lattice by Reading [74] and by Caspard, Le Conte de Poly-barbut, and Morvan [21]. A finite lattice is *congruence-uniform* (or *bounded*) if it can be constructed from a one-element lattice by a sequence of interval doublings; see Figure 1.2. This definition was shown to be equivalent to the usual one by Day [26], which we recall in §2.2.

Reading defined an edge-labeling called a *CN-labeling* whose existence is equivalent to the lattice being congruence-normal (see Theorem 2.2.2). One of our main results is to give some criteria for a collection of biclosed sets to form a congruence-uniform lattice, which we prove using CN-labelings (Theorem 3.1.9). These criteria are based on a formula for the join of two inversion sets for Coxeter groups giver by Dyer [30, Proposition 5.2]. In particular, we may deduce from Dyer's work that the (finite) intervals of the weak order of any Coxeter group is a congruence-uniform lattice.

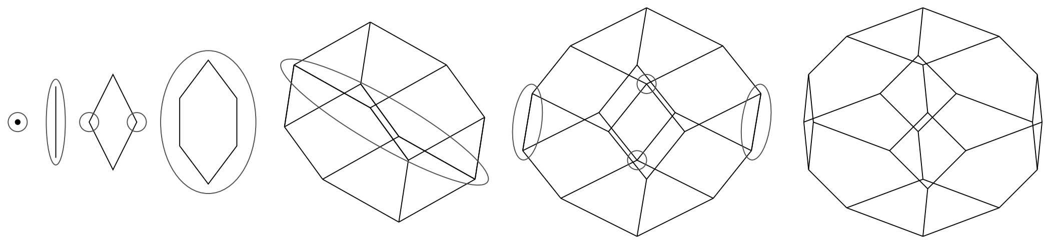


Figure 1.2: A sequence of doublings, ending with the weak order on $\mathfrak{S}_4$.

In §8.4, we consider a space of paths in a grid that take South and East steps. We say a collection X of paths is closed if whenever $p, q \in X$ and the concatenation of p and q is a path, the concatenation is in X. We prove that the set of biclosed collections of paths satisfies the conditions of Theorem 3.1.9, so it is a congruence-uniform lattice.

1.2 Lattice quotients

The *Tamari order* is a poset of bracketings of a word, ordered by a left-to-right associativity law; see §2.4.5 for a precise definition. Among the most significant features of the Tamari order is its geometric realizability by the associahedron and its lattice structure. Tamari orders and their generalizations have appeared in many parts of the literature. We recommend the book [96] for an introduction to many recent developments on these posets.

In their work on shellability of nonpure posets, Björner and Wachs defined a map from permutations of $[n]$ to bracketings of a word of length $n+1$ [17]. They introduced this map since it carries some of the topological structure of the order complex of the weak order to the Tamari order. Several other applications of this map may be found in [9], [62], and [97].

In [76], Reading proved that the permutations to bracketings map is a lattice quotient map. Consequently, the Tamari order inherits a congruence-uniform lattice structure from the weak order. Other proofs of the congruence-uniformity of the Tamari order appear in [22], [41].

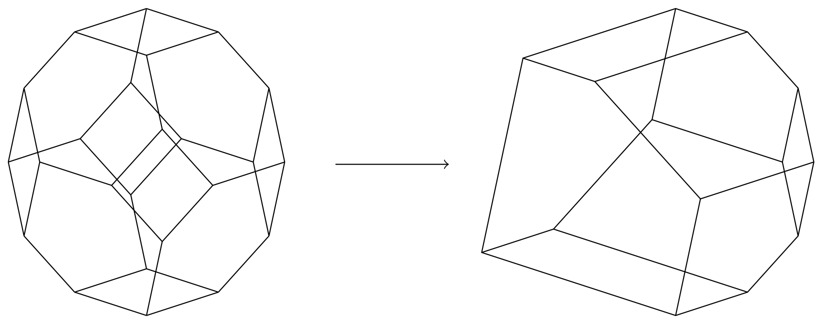


Figure 1.3: The standard map from the weak order to the Tamari order.

Many generalizations of the Tamari order have appeared in the literature such as the Higher Stasheff-Tamari orders [73], m-Tamari orders, and Cambrian lattices [77]. Santos, Stump, and Welker recently introduced a new generalization, the Grassmann-Tamari order $\mathrm{GT}_{k,n}$, which is an ordering on the facets of a "non-crossing" complex on $\binom{[n]}{k}$ [89]. The non-crossing complex is a certain flag, regular, unimodular, Gorenstein triangulation of an order polytope. From work of Sturmfels [93], the nice properties of this complex translate into nice algebraic properties of some associated rings. This complex also admits a nice combinatorial description, which we recall in §8.

In [89], Santos, Stump, and Welker conjecture that the Grassmann-Tamari orders are lattices. In §8, we make a further generalization to the *Grid-Tamari order*, which is an ordering on the facets of a "non-kissing" complex on SE-paths in a square grid pattern. We then prove that the Grid-Tamari orders are lattice quotients of the lattice of biclosed sets of paths described in the previous section, thus resolving the conjecture in [89]. Moreover, the Grid-Tamari order inherits the congruence-uniform structure from the lattice of biclosed sets of paths.

For any lattice L, the poset $\mathrm{Con}(L)$ of lattice congruences ordered by refinement is a distributive lattice. If L is a finite congruence-uniform lattice, the join-irreducibles of $\mathrm{Con}(L)$ are in natural bijection with the join-irreducibles of L. This fact allows us to give a very simple description of the collection of lattice congruences of Grid-Tamari orders; see Theorem 8.7.1. The join-irreducibles of the Grassmann-Tamari order $\mathrm{GT}_{3,6}$ is shown in Figure 1.4.

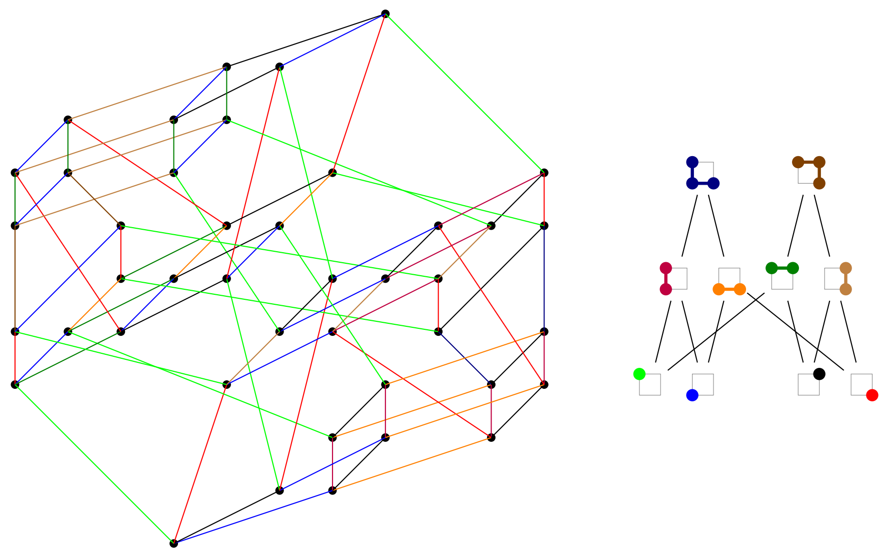


Figure 1.4: (left) $\mathrm{GT}_{3,6}$ (right) $\mathrm{JI}(\mathrm{Con}(\mathrm{GT}_{3,6}))^*$

1.3 Real Hyperplane Arrangements

The characterization of inversion sets of permutations as biclosed subsets of $\binom{[n]}{2}$ may be interpreted geometrically. A finite central *hyperplane arrangement* in $\mathbb{R}^n$ determines a complete fan of polyhedral cones, whose maximal cones are called *chambers*. A permutation σ of $[n]$ corresponds to the chamber

$$c_\sigma = \{\mathbf{x} \in \mathbb{R}^n : \ x_{\sigma(1)} < \cdots < x_{\sigma(n)}\}$$

of $\mathcal{A}^n_{braid}$, the arrangement of hyperplanes in $\mathbb{R}^n$ defined by the equations $x_i = x_j$ for $i \neq j$. The inversion set of a permutation σ may be identified with the *separation set* $S(c_\sigma)$, the set of hyperplanes separating c_σ from c_{id}.

A subset I of an arrangement $\mathcal{A}$ is *biclosed* with respect to a chamber c_0 if $I \cap \mathcal{A}'$ is the set of hyperplanes separating c_0 from some chamber of $\mathcal{A}'$ for every rank 2 subarrangement $\mathcal{A}'$. By the characterization of inversion sets, a subset I of $\mathcal{A}^n_{braid}$ is the separation set of some chamber if and only if I is biclosed with respect to c_{id}. More generally, the elements of a finite Coxeter group W correspond to biclosed subsets of its

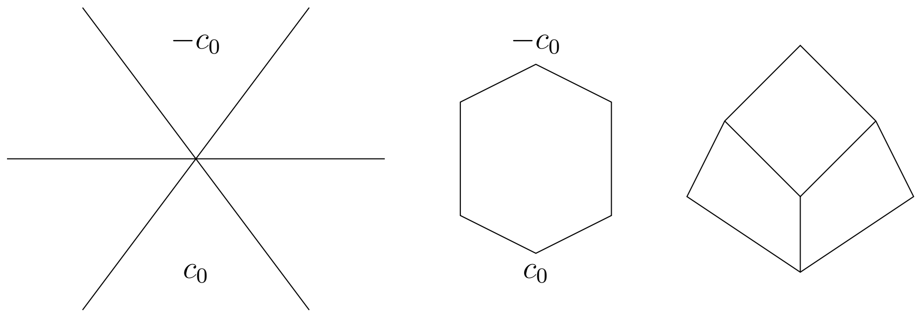


Figure 1.5: (left) An arrangement of three lines (center) Poset of chambers (right) Poset of 2-closed subsets.

standard reflection arrangement $\mathcal{A}(W)$ ([19]; see also the appendix of [70]). We prove this fact more generally for simplicial or supersolvable arrangements, which we recall in Sections 4.2 and 4.3.

A hyperplane arrangement $\mathcal{A}$ with a fundamental chamber determines a convex geometry on $\mathcal{A}$, which we recall in §3.2.3 (see [14, Proposition 5.1]). The chambers of $\mathcal{A}$ form a poset $\mathrm{Ch}(\mathcal{A}, c_0)$ where $c \leq c'$ if $S(c) \subseteq S(c')$; see Figure 1.5. If $\mathcal{A}$ is the standard reflection arrangement of a finite Coxeter group W, then this poset is isomorphic to the weak order on W. Björner, Edelman and Ziegler proved that if $\mathrm{Ch}(\mathcal{A}, c_0)$ is a lattice, then the chambers of $\mathcal{A}$ are in bijection with biconvex subsets of $\mathcal{A}$ [14, Theorem 5.5]. Furthermore, for chambers c and c' the separation set of $c \vee c'$ is the convex closure of $S(c) \cup S(c')$.

A subset I of an arrangement $\mathcal{A}$ with a fundamental chamber is *2-closed* if $I \cap \mathcal{A}'$ is convex for every rank 2 subarrangement $\mathcal{A}'$ of $\mathcal{A}$. By reduction to the rank 2 case, I is biclosed if and only if both I and $\mathcal{A} - I$ are 2-closed. The 2-closure is typically weaker than convex closure and seldom defines a convex geometry, even for reflection arrangements [70, Theorem 1(c)]. However, the two operators do agree for the reflection arrangements of types A or B [70, Theorem 1(b)], [92], where the convex closure may be interpreted as a transitive closure for posets or signed posets, respectively (see [82]). Stembridge determined the relative strength of various r-closures on reflection arrangements by computing their set of irreducible circuits, a special collection of circuits that suffices to compute the convex closure (see [92, Proposition 1.1] for a precise statement).

Even though the 2-closure is generally weaker than convex closure, Dyer proved for finite reflection arrangements that $S(c \vee c')$ is the 2-closure of $S(c) \cup S(c')$ for chambers c and c' ([30, Theorem 1.5]). We prove an analogue of this result for bineighborly and supersolvable arrangements in Theorems 5.5.1 and 4.3.2.

1.4 Reduced galleries

A *gallery* of a real central hyperplane arrangement $\mathcal{A}$ is a sequence of chambers $c_0, c_1, \ldots, c_m$ such that adjacent chambers are separated by exactly one hyperplane. The gallery is *reduced* if c_0 and c_m are separated by m hyperplanes. For any codimension 2 subspace $X \in L(\mathcal{A})$, a gallery between opposite chambers $c_0, -c_0$ can cross the hyperplanes containing X in two ways. Fixing a fundmental gallery r_0, we let $L_2(r)$ be the set of codimension 2 intersection subspaces on which r and r_0 disagree.

The set of reduced words of a Coxeter group element w may be identified with maximal chains of the interval $[e, w]$ of the weak order. A maximal chain in $\mathrm{Ch}(\mathcal{A}, c_0)$ induces a total order on $\mathcal{A}$. We call a total order on $\mathcal{A}$ *admissible* if its restriction to $\mathcal{A}'$ defines a reduced gallery for all rank 2 subarrangements $\mathcal{A}'$ of $\mathcal{A}$.

If W is a (possibly infinite) Coxeter group, then an admissible total ordering of its reflection arrangement $\mathcal{A}(W)$ is called a *reflection order* or *convex order*. When W is finite, there is a well-known correspondence between reflection orders and reduced words for the longest element. When W is infinite, the collections of biclosed sets and reflections orders are not completely understood; see [30] or [31] for conjectures and recent progress. If W is a Weyl group, a slightly different definition of convex order is used. In this setting, convex orders for affine Weyl groups were classified by Ito [53].

The set of reduced galleries between a fixed pair of chambers forms a graph where two galleries are adjacent if one gallery may be obtained from the other by "flipping" about a codimension 2 intersection subspace. The graph of reduced galleries was shown to be connected in successively greater generality by Tits [98], Deligne [28], Salvetti [87], and Cordovil-Moreira [24]. The graph of reduced galleries between opposite chambers has further topological connectivity (see [11]) as well as further graph-theoretic connectivity in some special cases (see [4] or [6]). Using a result from [4], we prove in Proposition 4.1.3 that a graph of reduced galleries of a bineighborly arrangement exhibits high graph-theoretic connectivity.

More recently, the diameter of some reduced gallery graphs of supersolvable arrangements were computed by Reiner and Roichman [81, Theorem 1.1] (see also [27]). Namely, if c_0 is incident to a modular flag of $\mathcal{A}$, then the graph of reduced galleries between c_0 and $-c_0$ has diameter equal to the number of codimension 2 intersection subspaces of $\mathcal{A}$. In particular, the graph of reduced words for the longest element in types A_n and B_n have diameter in $O(n^4)$. A key step in their proof relied on a (then) unproven assumption, that the chambers of a supersolvable arrangement correspond to biclosed sets. We give this correction in Theorem 4.3.4.

1.5 Crosscut-simplicial lattices

Posets often arise as the set of faces of a regular CW-complex ordered by inclusion. The original complex is recovered, up to homeomorphism, by taking the order complex $\Delta(P)$ of its poset of faces, where $\Delta(P)$ is the simplicial complex of chains $x_0 < \cdots < x_d$ in P. Hence, we refer to the topology of a poset as the topology of its order complex. Moreover, the Möbius invariant of $\hat{P}$ is equal to the reduced Euler characteristic of $\Delta(P)$.

The topology of open intervals of a poset P completely determines the *local topology* of $\Delta(P)$, the topology of links of faces of $\Delta(P)$. In many posets of combinatorial interest, every interval is either contractible or homotopy equivalent to a sphere. For example, the non-contractible intervals of the weak order are the "facial" intervals, which are homotopy equivalent to spheres [10]. Björner's proof uses the lattice property of the weak order and the Crosscut Theorem of Rota [86]. The key step is to show for any interval $[x, y]$ of the weak order, the join of any proper subset of atoms of (x, y) is not equal to y. We call a lattice *crosscut-simplicial* if it has this property.

The study of crosscut-simplicial lattices is intended as a lattice-theoretic companion to a combinatorial construction of Hersh and Mészáros [47]. They define an edge-labeling on a lattice called an *SB-labeling* whose existence implies the crosscut-simplicial property. For example, the Cayley graph labeling on the group of permutations $\mathfrak{S}_n$ generated by the simple transpositions is an SB-labeling of the weak order [47, Theorem 5.3].

Any congruence-uniform lattice admits a "canonical" CN-labeling, which we call a CU-labeling. In §5 we extend the definition of SB-labeling somewhat and prove that CU-labelings are examples of SB-labelings. On the other hand, not every congruence-normal lattice is crosscut-simplicial, so CN-labelings are not SB-labelings in general.

A large family of lattices that are crosscut-simplicial are the meet-semidistributive lattices, which we recall in §5. In fact, a partial converse is true: If P is a poset of chambers of some real central hyperplane arrangement, then P is crosscut-simplicial if and only if it is meet-semidistributive if and only if the arrangement is bineighborly (Theorem 5.5.1). However, there do exist lattices with an SB-labeling that are not meet-semidistributive. At present, we do not know whether every crosscut-simplicial lattice (or even every meet-semidistributive lattice) admits an SB-labeling. A special subfamily of meet-semidistributive lattices, namely join-distributive lattices, were shown to inherit an SB-labeling from an associated convex geometry by Henri Mühle [67].

1.6 Homotopy-facial posets

Fix an arrangement $\mathcal{A}$ with fundamental chamber c_0. Edelman and Walker proved that every interval of the chamber poset $\mathrm{Ch}(\mathcal{A}, c_0)$ is either contractible or homotopy equivalent to a sphere [36]. More precisely, they proved that if $[x, y]$ is the set of chambers incident to a face, then (x, y) is homotopy equivalent to a sphere. Otherwise, (x, y) is contractible.

The above theorem establishes an isomorphism between the poset of faces $\mathcal{L}(\mathcal{A})$ of $\mathcal{A}$ and the poset $\mathrm{Int}_{\mathrm{nonc}}(\mathrm{Ch}(\mathcal{A}, c_0))$ of noncontractible intervals of $\mathrm{Ch}(\mathcal{A}, c_0)$, ordered by inclusion. For an arbitrary bounded poset P, the full interval poset $\overline{\mathrm{Int}}(P)$ has a deformation retract to its subposet of proper noncontractible intervals. Furthermore, $\overline{\mathrm{Int}}(P)$ is homeomorphic to the suspension of $\overline{P}$. Thus, the theorem of Edelman and Walker may be viewed as an explanation of the homotopy equivalence

$$\overline{\mathcal{L}}(\mathcal{A}) \simeq \mathrm{susp}(\overline{\mathrm{Ch}}(\mathcal{A}, c_0)).$$

Two other families of posets conjectured to have a similar "homotopy-facial" property are gallery posets of supersolvable arrangements and Higher Bruhat orders. We briefly define these posets in §1.6.1 and §1.6.2.

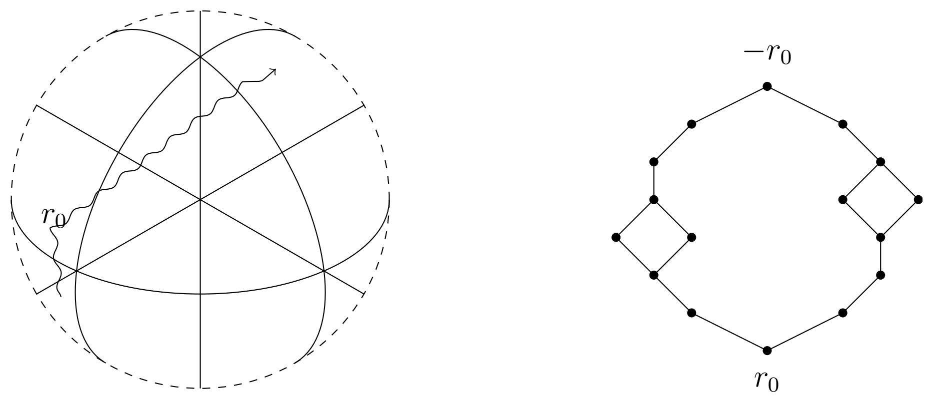


Figure 1.6: (left) A_3 reflection arrangement with fundamental gallery r_0 (right) $\mathrm{Gal}(\mathcal{A}, r_0)$

1.6.1 Gallery posets

In §7, we define a partial order $\mathrm{Gal}(\mathcal{A}, r_0)$ on the set of reduced galleries in an arrangement $\mathcal{A}$ with the same endpoints as a fixed gallery r_0. Gallery posets for general arrangements are not well-behaved. However, when $\mathcal{A}$ is supersolvable and r_0 is incident to a modular flag, we prove that $\overline{\mathrm{Gal}}(\mathcal{A}, r_0)$ is homotopy equivalent to a ($\mathrm{rk}\,\mathcal{A}-3$)-sphere using Rambau's Suspension Lemma. Moreover, if $[r, r']$ is the set of reduced galleries incident to a given cellular string, then (r, r') is homotopy equivalent to a sphere. We conjecture that all other intervals are contractible.

Reflection arrangements of type A or B are supersolvable. In this situation, we may identify $\mathrm{Gal}(\mathcal{A}, r_0)$ with a poset of reduced words for the longest element. Our result/conjecture on the topology of intervals of $\mathrm{Gal}(\mathcal{A}, r_0)$ seems to be new even in this special case.

The set $\omega(\mathcal{A}, c_0)$ of cellular strings with endpoints $c_0, -c_0$ is ordered by refinement. For supersolvable arrangements, our conjecture would establish a poset isomorphism between $\omega(\mathcal{A}, c_0)$ and the poset of non-contractible intervals of $\mathrm{Gal}(\mathcal{A}, r_0)$. By a general poset topology result, this in turn implies that $\omega(\mathcal{A}, c_0)$ is homotopy equivalent to the suspension of $\overline{\mathrm{Gal}}(\mathcal{A}, r_0)$. Hence, it is a homotopy sphere of dimension $\mathrm{rk}\,\mathcal{A}-2$. Actually, this fact was proved for arbitrary arrangements by Björner [11]. One may also deduce this fact from the Generalized Baues Theorem of Billera, Kapranov, and Sturmfels [7].

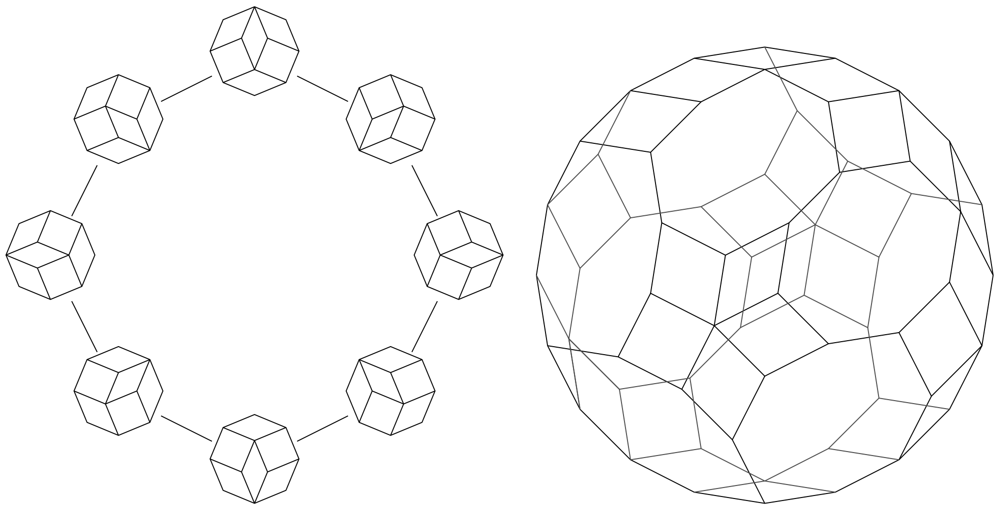


Figure 1.7: (left) HB$(4, 2)$ as a poset of rhombic tilings of a zonogon. (right) HB$(5, 2)$

Björner's proof has a wide-reaching generalization to the cellular string posets of duals of shellable CW-spheres [4].

1.6.2 Higher Bruhat orders

The Higher Bruhat order HB(n, d) is a collection of biclosed subsets of $\binom{[n]}{d}$, where X is closed if for $I \in \binom{[n]}{d-2}$ and $i, j, k \in [n] - I,\ i < j < k,\ I \cup \{i, j\} \in X$ and $I \cup \{j, k\} \in X$ implies $I \cup \{i, k\} \in X$. These posets were introduced by Manin and Schechtman to study a generalization of Yang-Baxter equations [63]. They appear in a variety of areas, including higher categories and Zamolodchikov's tetrahedron equations [55], soliton solutions of the Kadomtsev-Petviashvili equation [29], and the Multidimensional Cube Recurrence [46].

The elements of HB$(n, 2)$ may be identified with simple (labeled) pseudoline arrangements in the plane or with rhombic tilings of a zonogon with n zones. More generally, Ziegler proved that HB(n, d) may be identified with the set of generic single element extensions of the alternating matroid $\mathcal{M}_{\text{alt}}^{n,n-d}$ [102]. By oriented matroid duality HB(n, d) is in natural correspondence with simple single element liftings of the dual matroid $\mathcal{M}_{\text{alt}}^{n,d}$. The Bohne-Dress Theorem then provides a canonical bijection to the cubical tilings of a cyclic zonotope $Z(n, d)$ with n zones in $\mathbb{R}^d$ [18]. Through these bijections, the extension space conjecture of Sturmfels and Ziegler [94] translates into a special case of the Generalized Baues Problem posed by Billera, Kapranov, and Sturmfels [7].

The Generalized Baues Problem is to determine for a given projection of polytopes $\pi : P \to Q$ whether the set of π-induced subdivions of Q ordered by refinement is homotopy equivalent to a sphere of dimension $\dim P - \dim Q - 1$. It is well-known that the subposet of coherent subdivisions is polytopal and therefore homeomorphic to a sphere of dimension $\dim P - \dim Q - 1$ [8]. A stronger form of the Generalized Baues Problem asks whether the inclusion of this subposet into the full poset of subdivisions is a deformation retract, thus inducing a homotopy equivalence [83]. While some counterexamples are known, there are some cases in which the conjecture has been confirmed. Presently most of the known affirmative results are for low dimension/codimension or for very special polytopes P, Q [5],[3],[83].

The cubical tilings of $Z(n, d)$ are the minimal elements of the subdivision poset associated to the natural projection of the n-cube onto $Z(n, d)$. Thus, the higher Bruhat orders may be viewed as a particular way to order the minimal elements of this subdivision poset. The non-minimal elements correspond to non-contractible intervals of $\mathrm{HB}(n, d)$ in a natural way. It is conjectured by Reiner [83] that these intervals are the only non-contractible intervals of $\mathrm{HB}(n, d)$. This would give another proof that the poset of zonotopal tilings of a cyclic zonotope $Z(n, d)$ has the homotopy type of an $(n - d - 1)$-sphere. For $d = 1$, $\mathrm{HB}(n, 1)$ is isomorphic to the weak order on $\mathfrak{S}_n$, so Reiner's conjecture reduces to Björner's theorem on intervals of the weak order. Our main result on Higher Bruhat orders is a proof of the conjecture for $d = 2$ in §6.

1.7 Organization

In Chapter 2, we establish notation and cover background on posets, lattices, and real hyperplane arrangements necessary for our results. We also include a quick introduction to Coxeter groups to motivate the basic examples of biclosed sets at the end of the chapter. Following this, Chapter 3 includes many minor results used in later sections. The rest of the thesis is dedicated to proving the main results listed in the abstract.

Chapter 2

Background

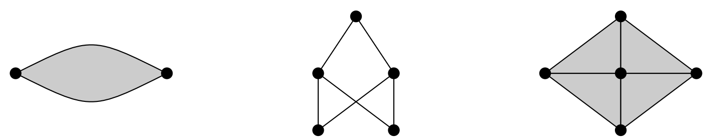
We establish some notation and fundamental results in this section. Most of the results in this section are well-known, so we state them without proof. In §2.1 we introduce notation for posets and order complexes. Topological methods for order complexes are reviewed later in §3.3. Lattices are discussed in §2.2. The most important lattice properties in the context of biclosed sets are semidistributivity and congruence-normality, which we discuss further in §3.1.3. Real hyperplane arrangements and oriented matroids are reviewed in §2.3. Specific techniques for arrangements are covered in §3.2.

2.1 Poset Topology

A *poset* $(P, \leq)$ is a set P with a reflexive, antisymmetric, and transitive relation $\leq$. If $x < y$, we say y *covers* x if there does not exist z such that $x < z < y$. We write $x \lessdot y$ if y covers x, and let $\mathrm{Cov}(P)$ denote the set of pairs (x, y) for which $x \lessdot y$. If P is a poset with an element x such that $x \leq y$ for all $y \in P$, then x is the *bottom* element of P, typically denoted $\hat{0}$. Dually, the *top* element of P is denoted $\hat{1}$. If P has a bottom or top element, the *proper part* $\overline{P}$ be the same poset with those bounds removed.

An *order ideal* X of a poset P is a subset of P such that if $x \leq y$ and $y \in X$ then $x \in X$. We let $\mathcal{J}(P)$ denote the set of order ideals of P. The *dual poset* P^* has the same underlying set as P where $x \leq_{P^*} y$ if and only if $y \leq_P x$.

Given $x \leq y$, the *closed* (*open*) interval $[x, y]$ ((x, y)) is the set of $z \in P$ such that $x \leq z \leq y$ ($x < z < y$). We let $P_{<x}$ ($P_{>x}$) denote the set of $y \in P$ for which $y < x$



$$\Sigma \longrightarrow P(\Sigma) \longrightarrow \Delta(P(\Sigma))$$

Figure 2.1: The order complex of the poset of faces of Σ is the barycentric subdivision of Σ.

$(y > x)$. Let $\mathrm{Int}(P)$ be the poset of closed intervals of P, ordered by inclusion. The *Möbius function* of a finite poset P is the unique function $\mu : \mathrm{Int}(P) \to \mathbb{Z}$ such that

$$\sum_{z \in [x,y]} \mu([x,z]) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise.} \end{cases}$$

The *Möbius invariant* of a bounded poset is $\mu(\hat{0}, \hat{1})$. One of the most significant uses for Möbius functions is the *Möbius inversion formula*: If f and g are functions from a finite poset P to some abelian group, then

$$f(y) = \sum_{x \le y} g(x) \quad \text{if and only if} \quad g(y) = \sum_{x \le y} \mu(x,y) f(x).$$

For example, if P is a chain poset, then the above formula is known as the Discrete Fundamental Theorem of Calculus. If P is a Boolean lattice, then Möbius inversion is equivalent to the Principle of Inclusion-Exclusion. When P is the divisor lattice, the formula reduces to classical Möbius inversion.

Let (Δ, A) be an abstract simplicial complex on the ground set A, and let $F \in \Delta$. Let $\|\Delta\|$ denote a topological space triangulated by Δ. The *deletion* $\mathrm{dl}_\Delta(F)$ of F is the subcomplex of Δ of faces disjoint from F. The *star* $\mathrm{st}_\Delta(F)$ of F is the subcomplex of faces F' such that $F \cup F' \in \Delta$. The *link* $\mathrm{lk}_\Delta(F)$ of F is the subcomplex of $\mathrm{st}_\Delta(F)$ of faces disjoint from F.

The *join* $\Delta * \Delta'$ of two complexes $(\Delta, A), (\Delta', A')$ is the simplicial complex on $A \sqcup A'$ with faces $F \sqcup F'$ where $F \in \Delta,\ F' \in \Delta'$. The join of abstract simplicial complexes realizes the topological join $\|\Delta * \Delta'\| \cong \|\Delta\| * \|\Delta'\|$. The *cone* $\{v\} * \Delta$ is the join of Δ with a one-element complex. The *suspension* $\{v, v'\} * \Delta$ is the join of Δ with a discrete two-element complex.

If F is a face of a simplicial complex Γ, the *stellation* of Γ at F, denoted $\mathrm{st}_F(\Gamma)$, is the simplicial complex $\mathrm{st}_F(\Gamma) = (\Gamma - F) \cup (\mathrm{lk}\, F * \partial F * \{v\})$ where v is a new vertex not in the ground set of Γ. When Γ is the boundary complex of a simplicial polytope P, the stellation at F may be geometrically realized by adding a new vertex to P "close" to the center of F.

The *order complex* $\Delta(P)$ of a poset P is the simplicial complex of chains $x_0 < \cdots < x_d$ of elements of P. If P is the set of faces of a regular CW-complex X ordered by inclusion, then the order complex of P is homeomorphic to X; see Figure 2.1. Thus, we define the topology of a poset to be that of its order complex. The link of a face $x_0 < \cdots < x_d$ is isomorphic to the join of the order complexes of $P_{<x_0}, (x_0, x_1), \ldots, (x_{d-1}, x_d), P_{>x_d}$. Hence, the local topology of P is completely determined by the topology of intervals and principal order ideals and filters of P.

If P is a bounded poset, the reduced Euler characteristic of $\overline{P}$ is equal to $\mu(\hat{0}, \hat{1})$. Hence, the Möbius invariant is a homotopy invariant of $\overline{P}$. The full Möbius function is then determined by the local topology of P. Many methods for computing homotopy invariants of posets are given in Section 10 of Björner's survey [12]. We review several relevant theorems in §3.3.

2.2 Lattices

Our notation for lattices mostly follows [43].

A *lattice* is a poset for which any two elements x, y have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. The elements $x \vee y$ and $x \wedge y$ are called *join* and *meet*, respectively. A *join-semilattice* (*meet-semilattice*) is a poset for which $x \vee y$ ($x \wedge y$) exists for any two elements x, y. A lattice is *complete* if the meet and join of an arbitrary collection of elements exists. We remark that finite lattices are automatically complete.

A lattice L is *meet-semidistributive* if $x \wedge z = y \wedge z$ implies $(x \vee y) \wedge z = x \wedge z$ for $x, y, z \in L$. A lattice is *join-semidistributive* if its dual is meet-semidistributive. A lattice is *semidistributive* if it is both meet- and join-semidistributive.

An equivalence relation Θ on a lattice L is a *lattice congruence* if $x \equiv y \mod \Theta$ implies $x \vee z \equiv y \vee z \mod \Theta$ and $x \wedge z \equiv y \wedge z \mod \Theta$ for $x, y, z \in L$. The set of equivalence classes L/Θ of a lattice congruence forms a lattice where $[x] \vee [y] = [x \vee y]$ and $[x] \wedge [y] = [x \wedge y]$ for $x, y \in L$. We say L/Θ is a *quotient lattice* of L, and the natural map $L \to L/\Theta$ is a *lattice quotient map*.

An element j of a lattice L is *join-irreducible* if $j \neq \hat{0}$ and for $x, y \in L$ such that $j = x \vee y$, either $j = x$ or $j = y$. If L is finite, j is join-irreducible exactly when it covers a unique element, which we call j_*. A *meet-irreducible* element m is defined dually and is covered by a unique element m^*. We let $\mathrm{JI}(L)$ and $\mathrm{MI}(L)$ denote the sets of join-irreducible and meet-irreducible elements of L, respectively.

A finite lattice is *Boolean* if it is isomorphic to the set of subsets of a set, ordered by inclusion. A lattice L is *join-distributive* if for all $x \in L,\ A \subseteq \{y : \ (x, y) \in \mathrm{Cov}(L)\}$ the interval $[x, \bigvee A]$ is a Boolean lattice. It is *meet-distributive* if its dual is join-distributive. A lattice is *distributive* if it is both join-distributive and meet-distributive. The operations of a distributive lattice L satisfy the distributive law: for $x, y, z \in L$,

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z),$$

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

A famous result of Birkhoff states that a finite lattice is distributive if and only if it is isomorphic to the poset of order ideals of its subposet of join-irreducibles.

Given a lattice L, its set of lattice congruences $\mathrm{Con}(L)$ forms a distributive lattice under refinement order. Hence when L is finite, $\mathrm{Con}(L)$ is isomorphic to $\mathcal{J}(\mathrm{JI}(\mathrm{Con}(L)))$. If y covers x, we write $\mathrm{con}(x, y)$ for the minimal lattice congruence in which $x \equiv y\ (\mathrm{con}(x, y))$ holds.

For any finite lattice L with lattice congruence Θ, we have

$$\Theta = \bigvee_{\substack{j \in \mathrm{JI}(L) \\ j \equiv j_* \mod \Theta}} \mathrm{con}(j_*, j).$$

Hence, the join-irreducible congruences are always of the form $\mathrm{con}(j_*, j)$ for some $j \in \mathrm{JI}(L)$. Conversely, if $(a, b) \in \mathrm{Cov}(L)$, then $\mathrm{con}(a, b)$ is a join-irreducible lattice congruence. A finite lattice L is *congruence-uniform* (or *bounded*) if

- the map $j \mapsto \mathrm{con}(j_*, j)$ is a bijection from $\mathrm{JI}(L)$ to $\mathrm{JI}(\mathrm{Con}(L))$, and

- the map $m \mapsto \mathrm{con}(m, m^*)$ is a bijection from $M(L)$ to $M(\mathrm{Con}(L))$.

Alternatively, finite congruence-uniform lattices may be characterized as homomorphic images of free lattices with bounded fibers or as lattices constructible from the one-element lattice by a sequence of interval doublings [26]. We describe the doubling construction below.

A subset C of a poset P is *order-convex* if $z \in C$ whenever $x, y \in C$ and $x \leq z \leq y$. Given an order-convex subset C of P, the doubling $P[C]$ is the induced subposet of $P \times \{0, 1\}$ with elements

$$P[C] = (P_{\leq C} \times \{0\}) \sqcup [(P - P_{\leq C}) \cup C] \times \{1\},$$

where $P_{\leq C} = \{x \in P : \ (\exists c \in C) \ x \leq c\}$. If P is a lattice, then $P[C]$ is a lattice where

$$(x, \epsilon) \vee (y, \epsilon') = \begin{cases} (x \vee y, \max(\epsilon, \epsilon')) & \text{if } x \vee y \in P_{\leq C} \\ (x \vee y, 1) & \text{otherwise} \end{cases},$$

for $(x, \epsilon), (y, \epsilon') \in P[C]$. A finite lattice L is *congruence-normal* if there exists a sequence of lattices $L_1, \ldots, L_l$ such that L_1 is the one-element lattice, $L_l = L$, and for all i, there exists an order convex subset C_i of L_i such that $L_{i+1} \cong L_i[C_i]$.

Theorem 2.2.1 *[Day [26]] Let L be a lattice. The following are equivalent.*

1. *The standard maps $\mathrm{JI}(L) \to \mathrm{JI}(\mathrm{Con}(L))$ and $M(L) \to M(\mathrm{Con}(L))$ are bijections.*

2. *L is a bounded lattice quotient of a free lattice.*

3. *L is congruence-normal and semidistributive.*

4. *There exists a sequence of lattices $L_1, \ldots, L_l$ such that L_1 is a 1-element lattice, $L_l = L$ and for each i, there exists a* closed interval *C_i such that $L_{i+1} \cong L_i[C_i]$.*

A finite congruence-uniform lattice is any lattice satisfying the conditions of Theorem 2.2.1. Another characterization of congruence-uniform lattices by edge-labeling is given in Theorem 2.2.3 below.

An *edge-labeling* of a poset P is a function from the covering relations $\mathrm{Cov}(P)$ to some label set R. Given a lattice L and poset R, an edge-labeling $\lambda : \mathrm{Cov}(L) \to R$ is a *CN-labeling* if L and its dual L^* both satisfy the following condition: For elements $x, y, z \in L$ with $(z, x), (z, y) \in \mathrm{Cov}(L)$ and maximal chains $C_1, C_2 \in [z, x \vee y]$ with $x \in C_1,\ y \in C_2$,

(CN1) the elements $x' \in C_1,\ y' \in C_2$ such that $(x', x \vee y), (y', x \vee y) \in \mathrm{Cov}(L)$ satisfy

$$\lambda(z, x) = \lambda(y', x \vee y),\ \lambda(z, y) = \lambda(x', x \vee y);$$

(CN2) if $(u, v) \in \mathrm{Cov}(C_1)$ with $z < u,\ v < x \vee y$, then $\lambda(z, x) \prec \lambda(u, v)$ and $\lambda(z, y) \prec \lambda(u, v)$; and

(CN3) the labels on $\mathrm{Cov}(C_1)$ are all distinct.

Theorem 2.2.2 ([74], Theorem 4) *A finite lattice L is congruence-normal if and only if it admits a CN-labeling.*

A *CU-labeling* $\lambda : \mathrm{Cov}(L) \to R$ is a CN-labeling for which both L and L^* satisfy

(CU) $\lambda((j_1)_*, j_1) \neq \lambda((j_2)_*, j_2)$ if $j_1, j_2 \in \mathrm{JI}(L),\ j_1 \neq j_2$.

A CU-labeling of the Grassmann-Tamari order $\mathrm{GT}_{3,6}$ is drawn in Figure 1.4. By a minor modification to Reading's proof of Theorem 2.2.2, one may prove the following.

Theorem 2.2.3 *A finite lattice L is congruence-uniform if and only if it admits a CU-labeling.*

A lattice is *polygonal* if for distinct elements x, y, z:

- if $z \lessdot x$ and $z \lessdot y$, then the interval $[z, x \vee y]$ contains exactly two maximal chains, and
- if $x \lessdot z$ and $y \lessdot z$, then the interval $[x \wedge y, z]$ contains exactly two maximal chains.

If L is a polygonal lattice, then intervals of the above form are called *polygons*. We note that the polygonal property is interesting from a combinatorial perspective: it implies the connectivity of the graph of maximal chains in an interval $[a, b]$, where two maximal chains are adjacent if they differ by a flip about a polygon. Polygonality also simplifies the description of lattice quotients, though we will not make use of this simplification. We refer to [79, §1-6] for more background on polygonal lattices.

2.3 Real hyperplane arrangements

In this section, we introduce some notation concerning hyperplane arrangements, oriented matroids, and polytopes. Some of the results we state without proof are surprisingly subtle. We recommend the books [15] and [103] for more background.

2.3.1 Basic definitions

Let V be a finite dimensional vector space. A *hyperplane* is a subspace of V of codimension 1. A hyperplane is *linear* if it contains the origin, and *affine* if it is a translate of a linear hyperplane. A *hyperplane arrangement* is a finite set of hyperplanes in V. In the literature, an arrangement of linear hyperplanes is said to be *central*, though we will assume our arrangements are central unless specified otherwise.

For a hyperplane arrangement $\mathcal{A}$, let $L(\mathcal{A})$ denote the collection of *intersection subspaces*, subspaces of the form $\bigcap_{H \in I} H$ where $I \subseteq \mathcal{A}$. Ordered by reverse-inclusion, the poset $L(\mathcal{A})$ is a *geometric lattice*, which means that it is atomic and upper-semimodular. Consequently, $L(\mathcal{A})$ is the lattice of flats of a simple matroid with ground set $\mathcal{A}$.

For our purposes, V will usually be a real vector space. Arrangements in real vector spaces determine a face lattice refining the intersection lattice as follows.

Let H be the (linear) hyperplane in $\mathbb{R}^n$ orthogonal to some vector v. Then H partitions $\mathbb{R}^n$ into three pieces H^0, H^+, H^-, where $H^0 = H,\ H^+ = \{w \in \mathbb{R}^n :\ v \cdot w > 0\}$ and $H^- = -H^+$; here, $v \cdot w$ is the usual inner product on $\mathbb{R}^n$. We generally assume H is equipped with an orientation without specifying a normal vector v.

If $\mathcal{A}$ is an arrangement of hyperplanes, then $\mathcal{A}$ divides $\mathbb{R}^n$ into a collection of relatively open cones of the form $\bigcap_{H \in \mathcal{A}} H^{x(H)}$ where $x \in \{0, +, -\}^{\mathcal{A}}$. Taking closures of these cones, $\mathcal{A}$ defines a complete fan $\mathcal{L}(\mathcal{A})$ of polyhedral cones. Cones in $\mathcal{L}(\mathcal{A})$ are called *faces* of $\mathcal{A}$.

A *sign vector* is an element of $\{0, +, -\}^{\mathcal{A}}$. If x is a sign vector for which the corresponding face $\bigcap_{H \in \mathcal{A}} H^{x(H)}$ is non-empty, then x is called a *covector* of $\mathcal{A}$. We use faces and covectors of arrangements interchangably. In particular, we also define $\mathcal{L}(\mathcal{A})$ to be the set of covectors of $\mathcal{A}$. For $x, y \in \{0, +, -\}^{\mathcal{A}}$, the *composite* $x \circ y$ is the sign vector where for $H \in \mathcal{A}$

$$(x \circ y)(H) = \begin{cases} x(H) & \text{if } x(H) \neq 0 \\ y(H) & \text{if } x(H) = 0 \end{cases}.$$

The set of sign vectors $\{0, +, -\}^{\mathcal{A}}$ is given the product order where $0 < +,\ 0 < -$, and $+$ is incomparable with $-$. For $x, y \in \{0, +, -\}^{\mathcal{A}}$, y is *incident* to x if $x \leq y$. We note that this abstract notion of incidence agrees with the usual notion for fans. A *circuit* $v \in \{0, +, -\}^{\mathcal{A}}$ is a minimal sign vector such that $\bigcap_{\substack{H \in \mathcal{A} \\ v(H) \neq 0}} H^{v(H)}$ is empty.

The intersection lattice and face semilattice are graded by codimension. We let $L_d(\mathcal{A})$ ($\mathcal{L}_d(\mathcal{A})$) be the set of codimension d intersection subspaces (faces) of $\mathcal{A}$. There is a natural map from $\mathcal{L}(\mathcal{A})$ to $L(\mathcal{A})$ respecting the grading defined by $x \mapsto \bigcap x^{-1}(0)$. In terms of matroids, this is the usual way of passing from an oriented matroid to its underlying matroid.

The set $\mathcal{L}(\mathcal{A})$ of covectors of an arrangement $\mathcal{A}$ satisfy

(L0) $\mathbf{0} \in \mathcal{L}$,

(L1) $x \in \mathcal{L}$ implies $-x \in \mathcal{L}$,

(L2) $x, y \in \mathcal{L}$ implies $x \circ y \in \mathcal{L}$, and

(L3) if $x, y \in \mathcal{L},\ H \in \mathcal{A}$ with $x(H) = -y(H)$, then there exists $z \in \mathcal{L}$ such that $z(H) = 0$ and $z(H') = (x \circ y)(H')$ for $H' \in \mathcal{A}$ with $(x \circ y)(H') = (y \circ x)(H')$.

For a finite set E, a subset of $\{0, +, -\}^E$ satisfying (L0)-(L3) is the set of covectors of an oriented matroid. For the most part, we will stick with the more familiar language of hyperplane arrangements, though some results require the use of oriented matroids. The Topological Representation Theorem of Folkman and Lawrence states that any oriented matroid can be realized by an arrangement of *pseudohyperplanes*, which roughly speaking is a collection of "piecewise-linear hyperplanes" satisfying some compatibility relations [39]. Hence, we do not lose "too much" generality by restricting our attention to oriented matroids coming from hyperplane arrangements.

2.3.2 Chambers and galleries

We let $\mathrm{Ch}(\mathcal{A})$ denote the set of *chambers*, the maximal faces of $\mathcal{A}$. The *walls* $\mathcal{W}(c)$ of a chamber c is the set of hyperplanes in $\mathcal{A}$ incident to c. Given two chambers $c, c' \in \mathrm{Ch}(\mathcal{A})$, the *separation set* $S(c, c')$ is the set of hyperplanes in $\mathcal{A}$ separating c and c'; that is, $H \in S(c, c')$ if $(c \circ c')(H) \neq (c' \circ c)(H)$. Given a chamber $c_0 \in \mathrm{Ch}(\mathcal{A})$, the *poset of chambers* $\mathrm{Ch}(\mathcal{A}, c_0)$ is an ordering on $\mathrm{Ch}(\mathcal{A})$ where $c \leq c'$ if $S(c_0, c) \subseteq S(c_0, c')$. The distinguished chamber c_0 is called the *fundamental chamber*. If a fundamental chamber c_0 is given, we let $S(c)$ denote the separation set $S(c_0, c)$.

For $X \in L(\mathcal{A})$, the *restriction* $\mathcal{A}^X$ is the arrangement

$$\{H \cap X : \ H \in \mathcal{A}, \ H \not\supseteq X\}$$

of hyperplanes in X. If $X \in L(\mathcal{A})$, the *localization* $\mathcal{A}_X$ is the subarrangement of hyperplanes containing X. For $\mathcal{A}' \subseteq \mathcal{A}$, the restriction $x|_{\mathcal{A}'}$ of a covector x of $\mathcal{A}$ to $\mathcal{A}'$ defines a surjective map $\mathcal{L}(\mathcal{A}) \to \mathcal{L}(\mathcal{A}')$. Restriction also defines a surjective, order-preserving map of chamber posets $\mathrm{Ch}(\mathcal{A}, c_0) \to \mathrm{Ch}(\mathcal{A}', (c_0)|_{\mathcal{A}'})$. For $X \in L(\mathcal{A})$ we let c_X denote $c|_{\mathcal{A}_X}$.

The geometry of a chamber is completely determined by its walls. We record this fact in the following lemma.

Lemma 2.3.1 *For $c \in \mathrm{Ch}(\mathcal{A})$, the face poset of c is isomorphic to the face poset of $c|_{\mathcal{W}(c)}$.*

A *gallery* of a real central hyperplane arrangement $\mathcal{A}$ is a sequence of chambers $(c_0, c_1, \ldots, c_m)$ such that adjacent chambers are separated by exactly one hyperplane. The gallery is *reduced* if c_0 and c_m are separated by m hyperplanes. We will assume that a gallery is reduced unless indicated otherwise. Hence, we view galleries as maximal chains in an interval $[c_0, c_m]$ of a chamber poset $\mathrm{Ch}(\mathcal{A}, c_0)$.

For $X \in L(\mathcal{A})$, if $r = (c_0, c_1, \ldots, -c_0)$, we define r_X to be the sequence

$$((c_0)_X, (c_1)_X, \ldots, (-c_0)_X)$$

with repetitions removed. If r is a gallery, then r_X is a gallery of $\mathcal{A}_X$, which we call the *localization* of r at X. The L_2*-separation set* $L_2(r, r')$ is $\{X \in L_2(\mathcal{A}) \mid r_X \neq r'_X\}$.

For a fixed base chamber c_0, a gallery is determined by the order in which the hyperplanes are crossed. The set of galleries between antipodal chambers admits a free involution $r \mapsto -r$ when $|\mathcal{A}| > 1$, where $-r$ is the gallery from c_0 to $-c_0$ which crosses the hyperplanes of $\mathcal{A}$ in the reverse order of r. Any gallery r is determined by its L_2-separation set $L_2(r_0, r)$ from some fixed gallery r_0. This follows since the relative order of any two hyperplanes H, H' in the total order on $\mathcal{A}$ induced by r is the same as their relative order in $r_{H\cap H'}$.

A gallery r from c to c' is *incident to a face* $x \in \mathcal{L}(\mathcal{A})$ if r contains the chambers $x \circ c$ and $x \circ c'$. If r is incident to x, then r must contain a sequence of chambers $c_1, \ldots, c_l$ such that $c_1 = x \circ c,\ c_l = x \circ c'$ and each c_i is incident to x. If $x \circ c' = x \circ (-c)$, then there exists a new gallery r' that agrees with r away from x containing the chambers $x \circ (-c_l), \ldots, x \circ (-c_1)$. If $x \in \mathcal{L}_2(\mathcal{A})$, then we say that r' is obtained from r by *flipping* about x. It is easy to show that $L_2(r, r') = \{x^0\}$. We prove the converse statement in Proposition 3.2.5, though this is folklore.

It is often preferable to define incidence to subspaces, so we say a gallery is *incident to a subspace* $X \in L(\mathcal{A})$ if it is incident to some face spanning X. We remark that a gallery may cross all of the hyperplanes in $\mathcal{A}_X$ consecutively without being incident to X. An example for an arrangement of four planes in $\mathbb{R}^3$ is given in the discussion after Proposition 3.2.6.

2.3.3 Simplicial arrangements

A *simplicial cone* is a polyhedral cone whose face poset is a Boolean lattice. An arrangement is *simplicial* if every chamber is a simplicial cone.

Our interest in simplicial arrangements comes from the observation that many statements about Coxeter groups can be reformulated to hold for any simplicial arrangement. One can often replace the group structure from a Coxeter group with the following incidence property of simplicial arrangements: If c is a simplicial chamber and $H_1, \ldots, H_l$ are walls of c, then c is incident to $H_1 \cap \cdots \cap H_l$.

Let $\mathcal{A}_{\mathbb{C}}$ be the complexification of $\mathcal{A}$. If $\mathcal{A}$ is a reflection arrangement in $\mathbb{R}^n$, then the space $\mathbb{C}^n - \mathcal{A}$ is a $K(\pi, 1)$ space whose fundamental group is the corresponding braid group. Deligne proved that simplicial arrangements also have the $K(\pi, 1)$ property [28]. Most arrangements do not have this property however. In particular, Edelman and

Reiner proved that not all free arrangements are $K(\pi, 1)$ [35].

Unlike simplicial or simple polytopes, simplicial arrangements are far from generic. Known examples of simplicial arrangements tend to have a lot of symmetry, (e.g. reflection arrangements). An old open problem is to construct all simplicial arrangements in rank 3; see [44] for a proposed list. A large portion of the proposed classification of rank 3 simplicial arrangements was recently found to coincide with the classification of rank 3 Weyl groupoids. Without going into detail, these arrangements "look like" several reflection arrangements of Weyl groups glued together. A complete classification of Weyl groupoids of arbitrary rank is given in [25], and the relationship to simplicial arrangements is described in [45].

2.3.4 Supersolvable arrangements

An intersection subspace $X \in L(\mathcal{A})$ of an arrangement $\mathcal{A}$ is *modular* if $X+Y$ is in $L(\mathcal{A})$ for all $Y \in L(\mathcal{A})$. A rank r arrangement $\mathcal{A}$ is *supersolvable* if its intersection lattice contains a *modular flag*, a maximal chain $X_0 < X_1 < \cdots < X_r$ where $X_i \in L_i(\mathcal{A})$ is modular. If $X \in L_{r-1}(\mathcal{A})$ for a rank r arrangement $\mathcal{A}$, then X is called a *modular line.*

Complex supersolvable arrangements are topologically significant as examples of $K(\pi, 1)$ spaces [37], and as *free* arrangements [68]. A combinatorial consequence of freeness is that the *characteristic polynomial*

$$\chi(t) = \sum_{X \in L(\mathcal{A})} \mu(\hat{0}, X) t^{\operatorname{codim} X}$$

factors into linear terms [91].

Most results about supersolvable arrangements are proved inductively by localization at a modular line. This approach to supersolvable arrangements is suggested by the following recursive characterization obtained by Björner, Edelman, and Ziegler.

Proposition 2.3.2 *(BEZ [14]) Every arrangement of rank at most 2 is supersolvable. An arrangement $\mathcal{A}$ of rank $r \geq 3$ is supersolvable if and only if it contains a modular line l such that the localization $\mathcal{A}_l$ is supersolvable.*

An extension of an arrangement $\mathcal{A}$ is a hyperplane arrangement containing $\mathcal{A}$. Although supersolvable arrangements are far from generic, any arrangement can be made supersolvable by adding enough hyperplanes. This is in stark contrast to the simplicial arrangements of the previous section.

Corollary 2.3.3 *Any arrangement admits a supersolvable extension of the same rank.*

Proof: Let $\mathcal{A}$ be an arbitrary arrangement of rank $r \geq 3$, and let $l \in L_{r-1}(\mathcal{A})$ be a line. Decompose $\mathcal{A}$ into a disjoint union $\mathcal{A} = \mathcal{A}_l \sqcup (\mathcal{A} \setminus \mathcal{A}_l)$. Let $\mathcal{A}_0$ be the union of $\mathcal{A}_l$ with the set of hyperplanes of the form $l + (H \cap H')$ for some pair of hyperplanes H, H' in $\mathcal{A} \setminus \mathcal{A}_l$. By the inductive hypothesis, $\mathcal{A}_0$ has a supersolvable extension $\widetilde{\mathcal{A}_0}$ of rank $r-1$. Then the disjoint union $\widetilde{\mathcal{A}_0} \sqcup (\mathcal{A} \setminus \mathcal{A}_l)$ is a rank r supersolvable arrangement by Theorem 2.3.2. ∎

2.4 Coxeter groups

The main results in this thesis were heavily influenced by Coxeter groups. In this section, we define Coxeter groups and describe a few of their interesting properties. We make no attempt to give a complete treatment of Coxeter groups, instead focusing on aspects most relevant to this thesis. Our perspective on Coxeter groups most closely matches that of [78]. For a more thorough introduction to the general theory of Coxeter groups, we recommend the books [1], [13], [52].

In §2.4.1, we define finite Coxeter groups as groups generated by reflections in Euclidean space and as groups with a certain presentation. The equivalence of these definitions underlies many interesting combinatorial properties of Coxeter groups. In particular, it gives two equivalent definitions of the weak order of a Coxeter group, generalizing the weak order on permutations; see §2.4.2. We define root systems in §2.4.3. Root systems provide a third definition of the weak order as a poset of biclosed sets. All of the nice lattice properties of the weak order on permutations hold for any finite Coxeter group, as we explain in §2.4.4. In §2.4.5, we identify some significant lattice quotients of the weak order called Cambrian lattices. Although we do not require much knowledge of Cambrian lattices, they motivated the identification of the Grid-Tamari order as a lattice quotient of a poset of biclosed sets (Theorem 8.6.11). Indeed, we prove that Cambrian lattices of type A are examples of Grid-Tamari orders in §8.8.

2.4.1 Reflection groups

A *finite real reflection group* W is a finite group of linear symmetries generated by a set of reflections on $\mathbb{R}^n$, such as a dihedral group or a symmetric group. Let $\mathcal{A}$ be the set of hyperplanes fixed by some reflection in W. The set of chambers of $\mathcal{A}$ admits an action by W as linear symmetries. This action is *simply transitive*, so after fixing a fundamental chamber c_0, there is a canonical bijection $W \to \mathrm{Ch}(\mathcal{A})$ taking an element w to $w \cdot c_0$.

Example 2.4.1 *The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by permuting coordinates. The permutations that act as reflections are precisely the transpositions. For $i \neq j$, the transposition (ij) fixes the hyperplane H_{ij} defined by the equation $x_i = x_j$. Let $\mathcal{A} = \{H_{ij} : 1 \leq i < j \leq n\}$. A permutation σ defines a chamber*

$$c_\sigma = \{x \in \mathbb{R}^n : x_{\sigma(1)} < \cdots < x_{\sigma(n)}\}.$$

Composing permutations right to left, $\mathfrak{S}_n$ acts on $\mathrm{Ch}(\mathcal{A})$ by $\pi(c_\sigma) = c_{\pi\circ\sigma}$. This action clearly carries the (left) regular representation. Each of the hyperplanes H_{ij} contains the 1-dimensional subspace l spanned by $(1, \ldots, 1)$. Hence, we generally draw this arrangement in the quotient space $\mathbb{R}^n/l$, as in Figures REF and REF.

A (possibly infinite) *Coxeter system* (W, S) is a group W and finite set S such that W admits a presentation of the form $\langle S |\ (st)^{m(s,t)},\ (s, t \in S)\rangle$ where $m(s, s) = 1$ for $s \in S$, and $m(s, t) \in \{2, 3, \ldots\} \cup \{+\infty\}$, $m(s, t) = m(t, s)$ for $s, t \in S$ with $s \neq t$. This presentation is encoded in the *Coxeter diagram*, a graph on S where two elements $s, t \in S$ are connected by an edge if $m(s, t) \geq 3$ with label $m(s, t)$ if $m(s, t) \geq 4$. We often just declare W to be a *Coxeter group* without specifying the distinguished generating set S. The *rank* of a Coxeter system is the cardinality of S.

If W is finite, then it admits a faithful representation $W \to \mathrm{GL}(\mathbb{R}^n)$ as a finite real reflection group. Conversely, if W is a finite real reflection group, then (W, S) is a Coxeter system where S is the set of reflections fixing a wall of the fundamental chamber c_0. This connection between the geometry of real reflection groups and the combinatorics of Coxeter systems has produced a wide array of beautiful results.

When W is infinite, it still admits a faithful representation $W \to \mathrm{GL}(\mathbb{R}^n)$ where the elements of S act as reflections with respect to some bilinear form distinct from the usual dot product. The set of hyperplanes fixed by some conjugate of an element of S triangulates an open cone in $\mathbb{R}^n$ called the *Tits cone*. However, outside the Tits cone, the chamber geometry breaks down.

Example 2.4.2 *Let $\mathcal{A}$ be the reflection arrangement for the symmetric group from Example 2.4.1. The walls of c_{id} are the hyperplanes $H_{i,i+1}$ for $1 \leq i < n$. Let S be the set of* adjacent transpositions $s_i = (i, i+1)$ *for $1 \leq i < n$. By the above correspondence, the symmetric group $\mathfrak{S}_n$ admits a presentation of the form*

$$\langle S | \ (s_i s_j)^{m(i,j)}, \ (i, j \in [n-1]) \rangle$$

where $m(i,j)$ is the order of $s_i s_j$ in $\mathfrak{S}_n$. A simple computation shows that $m(i, i+1) = 3$ for all i, and $m(i,j) = 2$ if $|i-j| \geq 2$. This Coxeter system is said to be of type A_{n-1}.

For $J \subseteq S$, the *standard parabolic subgroup* W_J is the subgroup of W generated by J. Using the reflection representation, one can show that the pair (W_J, J) is a Coxeter system. A Coxeter system (W, S) is *reducible* if there exists a nonempty set $J \subsetneq S$ such that $W \cong W_J \times W_{S-J}$. Otherwise (W, S) is *irreducible*. Remarkably, the finite irreducible Coxeter systems have a nice classification: four infinite families labeled A_n, B_n (or C_n), D_n, $I_2(m)$ and six exceptionals $E_6, E_7, E_8, F_4, H_3, H_4$. The subscript indicates the rank, and the letter matches the classification of finite dimensional complex semisimple Lie algebras (excluding type $H_3, H_4, I_2(m)$).

For a Coxeter system (W, S), we let $L(W)$ denote the set of *parabolic subgroups*, subgroups of the form wW_Jw^{-1} where $w \in W, \ J \subseteq S$. Let $\mathcal{L}(W)$ denote the set of left cosets of standard parabolic subgroups. Our notation reflects the following fact: if $\mathcal{A}$ is the reflection representation of a Coxeter group W, then $L(W)$ is isomorphic to the lattice of intersection subspaces of $\mathcal{A}$, and $\mathcal{L}(W)$ is isomorphic to the face lattice of $\mathcal{A}$.

Finite reflection arrangements in $\mathbb{R}^n$ are *simplicial*, which means that every chamber has exactly n walls. This follows from the fact that any hyperplane arrangement has at least one simplicial chamber and that W acts simply transitively on the chambers. The simplicial complex realized by $\mathcal{A}$ is called the *Coxeter complex*. Alternatively, the

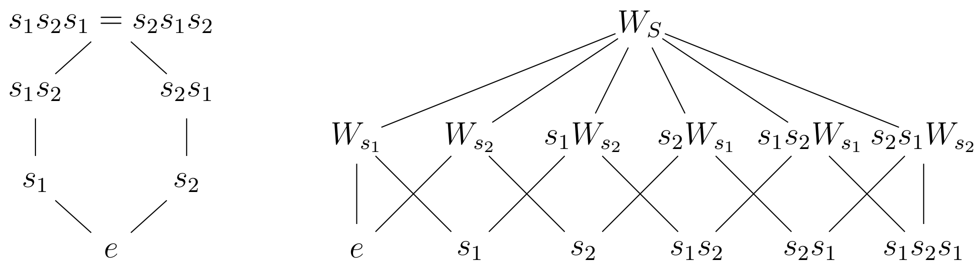


Figure 2.2: (Left) The weak order for the type A_2 Coxeter group. (Right) The parabolic cosets ordered by inclusion.

Coxeter complex may be defined abstractly as a simplicial complex on $\{wW_J : \ |J| = |S| - 1,\ w \in W\}$ with facets $\{wW_J : \ |J| = |S| - 1\}$ for $w \in W$.

2.4.2 Weak order

Let $T = \{wsw^{-1} \mid s \in S,\ w \in W\}$ denote the set of reflections of W. Define the *length* $l(w)$ of $w \in W$ to be the smallest value for which w equals $s_1 \cdots s_{l(w)}$ for some $s_i \in S$. The word $s_1 \cdots s_{l(w)}$ is called a *reduced word* for w. The (left) *inversion set* $\mathrm{Inv}(w)$ of $w \in W$ is $\{t \in T \mid l(tw) < l(w)\}$. The number of inversions of w is equal to its length. The (right) *descent set* $\mathrm{Des}(w)$ is $\{s \in S | \ l(w) > l(ws)\}$. Dually, the *ascent set* $\mathrm{Asc}(w)$ is $\{s \in S | \ l(w) < l(ws)\}$. The *weak order* is the ordering of W by inclusion of inversion sets; see Figure 2.2 or 2.3. Covering relations correspond to descents: for $u, v \in W$, $u \lessdot v$ if and only if there exists $s \in \mathrm{Des}(v)$ such that $u = vs$. Hence, if $u < v$ in the weak order, then there is a reduced word for v that extends a reduced word for u on the right.

Example 2.4.3 *Let (W, S) be the type A_{n-1} Coxeter system of Example 2.4.2. By our convention, the symmetric group acts on permutations on the left by permuting values, and it acts on the right by permuting positions. For example, if $w = 2314$ and $s = (12)$ then $ws = 3214$ and $sw = 1324$. Letting $s_i = (i, i+1)$, a reduced word for 2314 is s_1s_2.*

The inequality $2314 < 3421$ holds in the weak order since $\mathrm{Inv}(2314) = \{(12), (13)\}$ and $\mathrm{Inv}(3421) = \{(12), (13), (23), (14), (24)\}$. The permutation 3421 has a reduced word $s_1s_2s_1s_3s_2$, which extends a reduced word for 2314.

The weak order on the symmetric group admits some additional characterizations. Some of these interpretations are given in Figure 2.3.

Example 2.4.4 *Let (W,S) be a Coxeter system of rank 2, so that $|S| = 2$. Denote the elements of S by s, s', and let $m = m(s,s')$. There is a surjection from W to a dihedral group of order $2m$ taking s and s' to reflections. Using the realization of the dihedral group of order $2m$ as symmetries of a regular polygon with m sides, any word of length at most m with s and s' alternating (e.g. $e, s, s', ss', s's, ss's, \ldots$) is reduced. Moreover, the only two words that are expressions for the same element of W are the two alternating words of length m. It is easy to see that no other word is reduced; hence W is isomorphic to a dihedral group. This Coxeter system is said to be of type $I_2(m)$.*

The weak order has two maximal chains, which meet only at the top and bottom elements: $e < s < ss' < ss's < \cdots$ and $e < s' < s's < s'ss' < \cdots$. In the language of §2.2, this poset is a polygon.

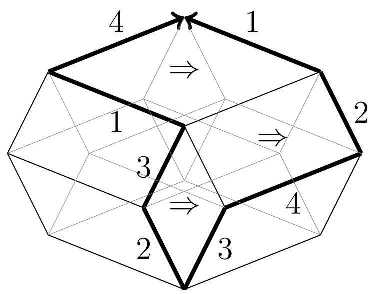
This example of a Coxeter group is significant as many results about Coxeter groups (and Lie algebras) may be proved by reduction to the rank 2 case. Reducing various problems to simpler problems in low rank is a major theme of this thesis.

Fix a fundamental chamber c_0 of the reflection arrangement $\mathcal{A}(W) = \{H_t : t \in T\}$ of a finite Coxeter group W. For $w \in W$, the separation set $S(c_0, w \cdot c_0)$ is equal to $\{H_t : t \in \mathrm{Inv}(w)\}$. The set of walls of $w \cdot c_0$ in $S(c_0, w \cdot c_0)$ is equal to $\{H_t : tw = ws,\ s \in \mathrm{Des}(w)\}$. Let w_0 be the (unique) element of W such that $w_0 \cdot c_0 = -c_0$. This element is called the *longest element* of W. From the chamber geometry, we may deduce that w_0 is the unique element for which $\mathrm{Inv}(w_0) = T$ and for which $\mathrm{Des}(w_0) = S$. If $J \subseteq S$ we let $w_0(J)$ be the longest element of (W_J, J).

Many nice properties of Coxeter groups follow from the fact that the weak order is a lattice, which we describe in detail in §2.4.4. The lattice structure of the weak order admits a nice local description. For $w \in W$, if $J \subseteq \mathrm{Des}(w)$, then $\bigwedge_{s \in J} ws = ww_0(J)$. Dually, if $J \subseteq \mathrm{Asc}(w)$, then $\bigvee_{s \in J} ws = ww_0(J)$. For a global description of meets and joins, we use a closure operator defined in the next section.

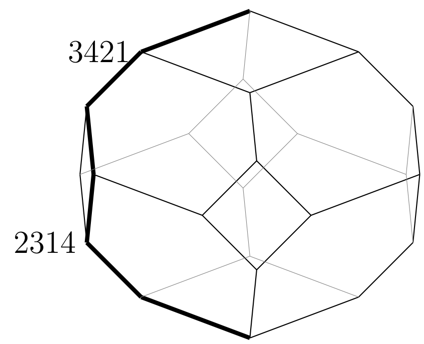
For $w \in W$, let $\mathcal{R}(w)$ be the set of reduced words for w. For $s, t \in S$, if $m(s,t) = m < \infty$ then $(st)^m = 1$. Moving half of the letters to the other side gives an equality

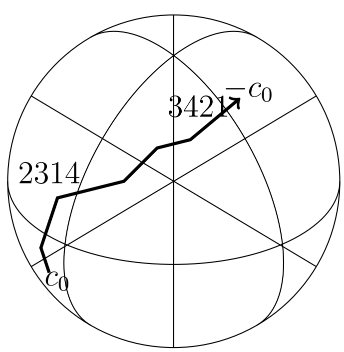
Figure 2.3: Several descriptions of the weak order on the symmetric group on n letters.



- Monotone paths in the cube;
- Maximal chains of the boolean lattice on $[n]$ ordered by diamond flips

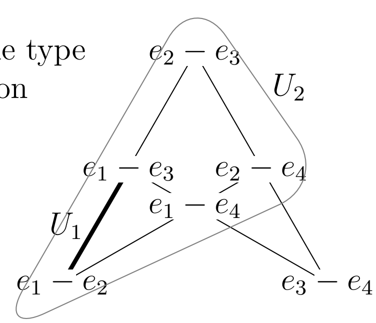
- The vertices of the permutahedron ordered by monotone paths;
- The vertices of the Cayley graph of the symmetric group $\mathfrak{S}_n$ with generators $(1,2), (2,3), \ldots, (n-1,n)$, ordered by geodesics from the identity permutation





- The chambers of the braid arrangement ordered by inclusion of separation sets from a base chamber

- Biclosed sets of positive roots in the type A_{n-1} root system, ordered by inclusion



$sts\cdots = tst\cdots$ between two words of length m. If $s_1\cdots s_l$ is a reduced word for w, then we can make a substitution of the form $sts\cdots \to tst\cdots$ to obtain a new reduced for w. Such moves are sometimes called *braid moves* or *elementary homotopies*.

If $w \in W$ has distinct descents (ascents) s, t, then the interval $[ww_0(W_{\{s,t\}}), w]$ ($[w, ww_0(W_{\{s,t\}})]$) is isomorphic to the weak order on $W_{\{s,t\}}$. As the weak order on any rank 2 Coxeter system is a polygon by Example 2.4.4, this implies that W is a polygonal lattice. The discussion of polygonal lattices in §2.2, suitably interpreted, then implies a famous result of Tits [98]: Given $w \in W$, the set $\mathcal{R}(w)$ is connected by braid moves.

2.4.3 Root systems

Separation sets (hence, inversion sets) of a reflection arrangement $\mathcal{A}$ admit a simple characterization: For $S \subseteq \mathcal{A}$, if $S \cap \mathcal{A}'$ is of the form $S(c_0|_{\mathcal{A}'}, c)$ for some chamber c in $\mathcal{A}'$ for any rank 2 subarrangement $\mathcal{A}'$, then $S = S(c_0, c)$ for some chamber c of $\mathcal{A}$. This characterization can be rephrased in terms of biclosed sets. Say $X \subseteq \mathcal{A}$ is *rank 2 convex closed* (or *2-closed*) if whenever three hyperplanes H_1, H_2, H_3 intersect at a common codimension 2 subspace X and $(c_0)_X$ is incident to H_1 and H_2, then $H_1, H_2 \in X$ implies $H_3 \in X$. Then a subset of $\mathcal{A}$ is biclosed if and only if it is a separation set. For general arrangements, separation sets are biclosed, but the converse may not hold. This characterization of separation sets in reflection arrangements is typically stated in terms of root systems as defined below.

Given a vector $\alpha \in \mathbb{R}^n$, let r_α be the reflection $r_\alpha(v) = v - \frac{2\alpha\cdot v}{\alpha\cdot\alpha}\alpha$, where $w \cdot v$ is the standard inner product. A *finite root system* is a finite subset Φ of $\mathbb{R}^n$ such that

- $0 \notin \Phi,\ \Phi \neq \emptyset$,
- Φ is closed under reflections r_α for $\alpha \in \Phi$,
- $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for $\alpha \in \Phi$

Some authors assume that Φ spans $\mathbb{R}^n$, but we do not require this. If $\mathbb{Z}\Phi$ is $\mathbb{Z}$-sublattice of $\mathbb{R}^n$, then Φ is said to be *crystallographic*.

If Φ is a root system, the group generated by $\{r_\alpha :\ \alpha \in \Phi\}$ is a finite reflection group. Conversely, given a finite reflection group W with reflection arrangement $\mathcal{A}$, any collection of vectors $\{\pm\alpha_H :\ H \in \mathcal{A}\}$ with $\alpha_H \in H^\perp$ is a root system if $\|\alpha_H\| = \|\alpha_{H'}\|$

whenever there exists $w \in W$ with $wH = H'$. If W is an irreducible Coxeter system not of type $H_3, H_4, I_2(m),\ m \neq 2, 3, 4, 6$, then the root system may be chosen to be crystallographic.

Example 2.4.5 *Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Let $\Phi = \{\alpha_{ij} :\ i \neq j\}$ where $\alpha_{ij} = e_i - e_j$. Then $r_{\alpha_{ij}}$ acts on $\mathbb{R}^n$ by swapping the i and j coordinates. It is easy to check that Φ satisfies the hypotheses of a root system. Moreover, $\{r_\alpha :\ \alpha \in \Phi\}$ generates the symmetric group action on $\mathbb{R}^n$ described in Example 2.4.2. As all transpositions in the symmetric group are conjugate, Φ is the unique root system for the symmetric group (up to orthogonal transformation and scaling).*

Fix a fundamental chamber c_0 for a reflection arrangement $\mathcal{A}$, and let Φ be an associated root system. If $v_0 \in \mathbb{R}^n$ is in the interior of c_0, then Φ decomposes as the union $\Phi^+ \sqcup \Phi^-$ where $\Phi^+ = \{\alpha \in \Phi :\ \alpha \cdot v_0 > 0\}$ and $\Phi^- = -\Phi^+$. The set Φ^+ is the set of *positive roots* (with respect to c_0). For each $t \in T$ let α_t be the corresponding positive root. If $w \in W$, then the inversion set of w is the set of reflections $t \in T$ for which $w(\alpha_t) \in \Phi^-$.

Say a subset X of Φ^+ is *2-closed* if $\alpha, \beta \in X$ and $\lambda\alpha + \mu\beta \in \Phi^+$ for some $\lambda, \mu \in \mathbb{R}_{>0}$, then $\lambda\alpha + \mu\beta \in X$. Translating between root systems and reflection arrangements, this closure is the same as the closure described in the beginning of this section.

Example 2.4.6 *Let Φ be the root system of Example 2.4.5. Let c_0 be the polyhedral cone defined by $x_1 \leq x_2 \leq \cdots \leq x_n$. Then c_0 is a chamber of $\mathcal{A}$. It defines a system of positive roots $\Phi^+ = \{e_j - e_i :\ i < j\}$. To a subset X of Φ^+ we associate the relation $P(X) = \{(i, j) :\ e_j - e_i \in X\}$. It is clear that this relation is acyclic, in the sense that there do not exist $a_1, \ldots, a_l$ such that $(a_1, a_2), \ldots, (a_l, a_1) \in P(X)$. Hence, its transitive closure defines a poset on $[n]$ for which $1, 2, \ldots, n$ is a linear extension. Such a poset is said to be* naturally labeled.

Recall that a relation P is transitive if $(i, j), (j, k) \in P$ implies $(i, k) \in P$. Hence, if $X \subseteq \Phi^+$ and $P(X)$ is transitive, then $e_j - e_i,\ e_k - e_j \in X$ implies $e_k - e_i \in X$ for $i < j < k$. As $e_k - e_i = (e_k - e_j) + (e_j - e_i)$, this condition is equivalent to X being 2-closed.

The equivalence between these two closure operators is a part of the cone-preposet dictionary *(see [71]). Under this correspondence, biclosed sets are in bijection with naturally labeled posets of order-dimension 2 [16].*

Let Φ^+ be a set of positive roots of some root system Φ. Say a subset X of Φ^+ is *convex closed* (or *convex*) if $\mathbb{R}_{\geq 0}X \cap \Phi^+ = X$. If X is biclosed with respect to the convex closure, we say X is *biconvex*. Since separation sets are biconvex and biconvex sets are biclosed in any acyclic vector configuration, these three classes coincide for root systems.

If Φ is the type A_{n-1} root system of Example 2.4.5, then the convex closure and the 2-closure are the same operator. If Φ is a type B_n root system, then these two closures again coincide. In this case, a closed set of positive roots is called a *signed poset*. Signed posets have many nice properties analogous to properties of ordinary posets. Unfortunately, for general root systems (e.g. D_4) the convex closure and 2-closure disagree [70],[92]. For this reason, it is often not clear how to appropriately define posets in other types.

2.4.4 Lattice structure

The lattice structure of the weak order on a finite Coxeter group is nicely described in terms of positive roots. The join of two elements $u, v \in W$ is the unique element $w \in W$ for which $\mathrm{Inv}(w)$ is the 2-closure of $\mathrm{Inv}(u) \cup \mathrm{Inv}(v)$. Dyer proved the following refinement of this fact: if $u, v, x \in W$ such that $x \leq u$ and $x \leq v$, then $\mathrm{Inv}(u \vee v)$ is the 2-closure of $(\mathrm{Inv}(u) \cup \mathrm{Inv}(v)) - \mathrm{Inv}(x)$ [30]. We prove more generally that the analogous statement holds for any bineighborly arrangement in §4.2.

The semidistributivity of the weak order follows from Theorem 3.1.7 as a consequence of Dyer's formula. Other proofs of semidistributivity of the weak order were given in [21], [59], [74].

Since the weak order is semidistributive, its elements admit canonical join representations. For $w \in W$, its canonical join representation is [80]

$$w = \bigvee_{s \in \mathrm{Des}(w)} \min\{u \in W : \ u \leq w, \ wsw^{-1} \in \mathrm{Inv}(u)\}.$$

We prove that semidistributive lattices are crosscut-simplicial in §5.3. For the weak order, this fact may be deduced from the following special case of Dyer's formula. For $w \in W$, if $J \subseteq \mathrm{Asc}(w)$ then $\bigvee_{s \in J} ws = ww_0(J)$. As $w_0(J) \neq w_0(J')$ whenever $J \neq J'$, it follows that the weak order is crosscut-simplicial. Moreover, the non-contractible intervals are exactly the parabolic cosets $\mathcal{L}(W)$ of W. This argument was originally given by Björner [10].

Polygonality of the weak order may be deduced from Theorem 3.1.11. Nathan Reading gave an alternative proof by showing that a chamber poset of a hyperplane arrangement is polygonal if and only if it is "tight" (or bineighborly) [79, Theorem 1-6.10].

If (W, S) is *simply-laced* (so $m(s, s') \leq 3$ for all $s, s' \in S$), then all of its rank 2 subsystems have either 2 or 3 reflections. We partially order the elements of T as follows. Let $\{t_1, t_2, t_3\}$ be the set of reflections of a rank 2 subsystem with 3 reflections. Let α_i be the positive root corresponding to t_i for $i \in [3]$. Then one of the roots is a positive sum of the other two, say $\alpha_3 = \alpha_1 + \alpha_2$. Order $t_1 \prec t_3$ and $t_2 \prec t_3$. The transitive closure of $\prec$ on T is called the *root poset* of W. Using this order on T, Theorem 3.1.9 implies that the weak order on W is congruence-uniform.

Actually, the weak order on any finite Coxeter system is congruence-uniform [74], [21]. Reading's proof replaces the root poset with an ordering of pieces of hyperplanes called *shards*. Given that the weak order is semidistributive, it is enough to show that the ordering on shards is acyclic.

2.4.5 Cambrian lattices

The *Tamari order* is a poset of bracketings of a word, ordered by a left-to-right associativity law. Formally, a *bracketing* of a word $a_1 \ldots a_n$ may be defined recursively as either the letter a_1 if $n = 1$ or an ordered pair of two bracketings on $a_1 \ldots a_i$ and $a_{i+1} \ldots a_n$ for some $i < n$. For example, $(a(bc))d$ represents the pair of bracketings $((a(bc)),\ d)$. Given bracketings A, B, C, we define a directed edge $(AB)C \to A(BC)$. This defines an acyclic directed graph on the set of bracketings of a fixed word. The Tamari order is the transitive closure of this relation; that is, $X \leq Y$ if there exists a sequence of bracketings $X = X_0, \ldots, X_t = Y$ such that $X_i \to X_{i+1}$ for all i.

This partial order was originally defined by Dov Tamari in his thesis as part of his work on sets with substitution rules. For example, the aforementioned directed associativity law may be viewed as a substitution rule. Tamari conjectured in his thesis that this poset is a lattice and gave a proof in [95]. Further proofs of the lattice property appear in [40], [50], and [51]. More recently, Geyer proved that the Tamari lattice is congruence-uniform [41]. Other proofs of congruence-uniformity appear in [22] and [76]. Reading's approach is to identify the Tamari order as a lattice quotient of the weak order. As taking lattice quotients preserves the lattice properties of §2.4.4, one may deduce that the Tamari order also has these properties.

The standard map from permutations of $[n]$ to bracketings of the word $a_0 \ldots a_{n+1}$ realizing the Tamari order as a lattice quotient of the weak order may be defined recursively as follows: Given a permutation $\sigma_1 \cdots \sigma_n$, for each $i \in [n]$, bracket the subword $a_j \cdots a_k$ such that $a_j < \sigma_i < a_k$, $\{a_j, a_k\} \cap \{\sigma_1, \ldots, \sigma_i\} = \emptyset$, and $a \in \{\sigma_1, \ldots, \sigma_i\}$ if $a_j < a < a_k$. We define a generalization of this map in §8.6.

Since the weak order is congruence-uniform, its set of lattice congruences is isomorphic to the lattice of order ideals of some poset on the join-irreducibles of the weak order. Reading identified a collection of particularly interesting lattice congruences called *Cambrian congruences*. The associated quotient lattices are called *Cambrian lattices*. A *Cambrian congruence* Θ is a lattice congruence generated by the following relations: for $s, s' \in S$, if $m = m(s, s') \geq 3$ then either

- $s_1 \equiv s_1 \cdots s_{m-1} \mod \Theta$, or
- $s_{2m} \equiv s_{2m} \cdots s_{m+2} \mod \Theta$,

but not both, where $s_i = s$ if i is odd and $s_i = s'$ if i is even. These generating relations may be encoded by an orientation of the Coxeter diagram, where $s' \to s$ means $s_1 \equiv s_1 \cdots s_{m-1} \mod \Theta$ where $s_i = s$ if i is odd and $s_i = s'$ if i is even.

Let (W, S) be a finite Coxeter system, and fix an orientation of its Coxeter diagram. From the classification, the Coxeter diagram of (W, S) is a forest, so any of its orientations is acyclic. Given an orientation of the Coxeter diagram, we say a total order $s_1, \ldots, s_r$ of the elements of S is compatible with the orientation if $s_i \to s_j$ implies $i < j$. The product $s_1 \cdots s_r$ in W is called a *Coxeter element*, usually denoted c. Although an oriented Coxeter diagram does not uniquely specify a compatible order on S, it does uniquely specify a Coxeter element. Conversely, if c is a Coxeter element, then any reduced word for c induces an orientation of the Coxeter diagram.

Let $c = s_1 \cdots s_r$ be a Coxeter element of W. An element $w \in W$ is *c-sortable* (or *sortable*) if there exists a reduced word $q_1 \cdots q_l$ for w ($q_i \in S$) and words $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)}$ such that

- the concatenation $\mathbf{q}^{(1)} \cdots \mathbf{q}^{(m)}$ equals $q_1 \cdots q_l$,
- $\mathbf{q}^{(1)}$ is a subword of $s_1 \cdots s_r$, and
- $\mathbf{q}^{(i+1)}$ is a subword of $\mathbf{q}^{(i)}$ for all i.

A fundamental result is that an element w is c-sortable if and only if it is the minimum element of its Θ-congruence class. Hence, any Cambrian lattice is isomorphic (as a join-semilattice) to the restriction of the weak order to c-sortable elements. In fact, one can show that the meet of any two c-sortable elements is c-sortable, so the Cambrian lattice is a sublattice of the weak order.

As a lattice quotient of the weak order, a Cambrian lattice may be realized as a poset of maximal cones of a complete fan coarsening the reflection arrangement [75]. We refer to this fan as a *c-Cambrian fan*. If c and c' are distinct Coxeter elements of the same Coxeter system (W, S), the c-Cambrian fan may be geometrically distinct from the c'-Cambrian fan in the sense that they are linearly non-isomorphic. Remarkably, the Cambrian fans for (W, S) are all combinatorially isomorphic, which means that they have isomorphic face posets. Indeed, the c-Cambrian fan is combinatorially isomorphic to the normal fan of a *generalized associahedron*, a polytope only depending on W. Moreover, for any Coxeter element c there is a realization of the generalized associahedron whose normal fan is a c-Cambrian fan. In particular, any two Cambrian lattices for the same Coxeter system have the same number of elements, called the W-Catalan number.

Generalized associahedra are certain simple polytopes that depend on W. The boundary complex of the polar dual is a flag, simplicial complex that can be defined in a variety of ways. For example, in type A, this complex is isomorphic to the abstract simplicial complex on the interior diagonals of a convex polygon, whose facets are triangulations. By polar duality, a Cambrian lattice is some ordering of the facets of this complex.

As a quotient lattice of the weak order, a Cambrian lattice is semidistributive, crosscut-simplicial, congruence-uniform, and polygonal. These lattice properties yield new insights on the combinatorics of generalized associahedra and related objects.

2.5 Biclosed sets: first examples

One interpretation of the weak order on a Coxeter group was given in terms of biclosed sets in §2.4.3. We give some fundamental examples of other collections of biclosed sets in this section.

2.5.1 Closure operators

A *closure operator* on a set S is an operator $X \mapsto \overline{X}$ on subsets of S such that for $X, Y \subseteq S$,

$$\begin{aligned} &X \subseteq \overline{X}, \\ &\overline{\overline{X}} = \overline{X}, \text{ and} \\ &X \subseteq Y \text{ implies } \overline{X} \subseteq \overline{Y}. \end{aligned}$$

In addition, we assume $\overline{\emptyset} = \emptyset$. A subset X of S is *closed* if $X = \overline{X}$. A set X is *co-closed* (or *open*) if $S - X$ is closed. We say X is *biclosed* (or *clopen*) if X and $S - X$ are both closed. We let $\mathrm{Bic}(S)$ be the poset of biclosed subsets of S ordered by inclusion. By our assumption, S and $\emptyset$ are always biclosed.

The poset $\mathrm{Bic}(S)$ is not necessarily a lattice. However, in many situations, the join of two biclosed sets is equal to the closure of their union. By adding some additional hypotheses, one may conclude that $\mathrm{Bic}(S)$ is semidistributive or congruence-normal; see §3.1.3.

A collection $\mathcal{B}$ of subsets of S is *ordered by single-step inclusion* if for all $X, Y \in \mathcal{B}$ such that $X \subsetneq Y$ there exists $y \in Y - X$ such that $X \cup \{y\} \in \mathcal{B}$. If $\emptyset, S \in \mathcal{B}$ and $\mathcal{B}$ is ordered by single-step inclusion, then it is a graded poset with rank function $X \mapsto |X|$ for $X \in \mathcal{B}$; in particular, every maximal chain has length $|S|$. For example, the collection of inversion sets of elements of a Coxeter group is ordered by single-step inclusion.

We may also use the phrase "ordered by single-step inclusion" to *define* an ordering on a family of sets. For example, let $S = \{1, 2, 3\}$ and let $\mathcal{B} = \{\emptyset, 1, 3, 12, 123\}$. Under (full) inclusion order, $3 < 123$, but 3 and 123 are incomparable under single-step inclusion order since there is no 2-element set between them. Single-step inclusion order is often more natural for posets defined by "local moves".

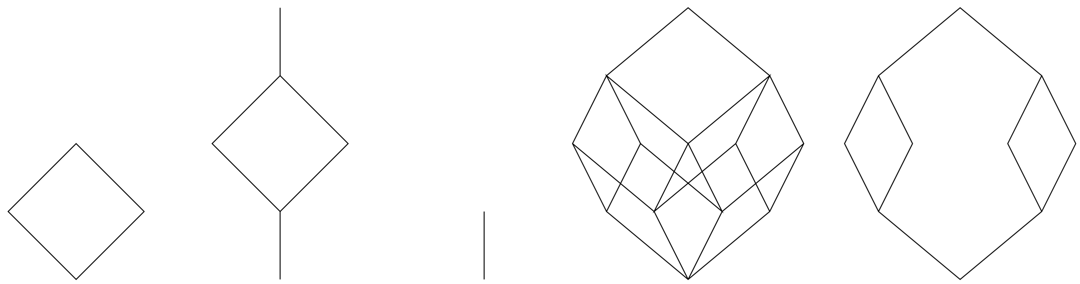


Figure 2.4: The closed and biclosed subsets of P with respect to the ideal closure and order-convex closure.

2.5.2 Closure operators on posets

Fix a finite poset P. Two natural closure operators on P are the ideal closure and order-convex closure, defined below. An example on a 4-element poset is given in Figure 2.4.

Given a subset X of P, the *ideal closure* of X is the set

$$\mathcal{J}(X) = \{y \in P : \ \exists x \in X, \ y \leq x\}.$$

Then a subset of P is closed (co-closed) exactly when it is an order ideal (filter). Thus, the closed subsets of P forms a distributive lattice. Conversely, any finite distributive lattice arises in this manner. A subset of P is biclosed if it is a union of connected components of P. Hence, $\mathrm{Bic}_{\mathcal{J}}(P)$ is a Boolean lattice on the set of connected components of P. The set $\mathrm{Bic}_{\mathcal{J}}(P)$ is ordered by single-step inclusion only if P is an antichain poset.

Given a subset X of P, the *order-convex closure* of X is the set

$$\mathrm{OConv}(X) = \{z \in P : \ \exists x, y \in X, \ x \leq z \leq y\}.$$

Closed subsets of P are said to be *order-convex*. Order-convex subsets are used to define the doubling operation on posets; see §2.2. If P has a bottom *or* a top element, then a subset is biclosed if and only if it is either an ideal or an order filter. Using this correspondence, the poset $\mathrm{Bic}_{\mathrm{OConv}}(P)$ is ordered by single-step inclusion.

Some interesting closure operators arise as an intersection of several order-convex closures. The intersection is defined in general as follows. Let A, B be subsets of a set S with closure operators $\mathrm{cl}_A, \mathrm{cl}_B$ on A and B. Extend cl_A and cl_B to S so that for $X \subseteq S$, $\mathrm{cl}_A(X) = X \cup \mathrm{cl}_A(A \cap X)$ and $\mathrm{cl}_B(X) = X \cup \mathrm{cl}_B(B \cap X)$. These operators induce a closure operator on S, where $\mathrm{cl}_S(X) = \bigcup_{m=1}^{\infty}(\mathrm{cl}_A \circ \mathrm{cl}_B)^m(X)$. Alternatively, $\mathrm{cl}_S(X)$ may be defined as the smallest subset of S containing X that is closed in both A and B. In general, if $\mathcal{B}$ is a family of subsets of S, each with their own closure operator, we define a closure on S where X is closed if $X \cap B$ is closed for all $B \in \mathcal{B}$.

2.5.3 Closure operators on real vector configurations

Fix a finite set E of vectors in $\mathbb{R}^n$. We assume the configuration E is *acyclic*, which means that there is no positive linear dependence relation among the vectors in E. Most of the closure operators we consider come from the following two examples. These closures are defined for root systems in §2.4.

For $X \subseteq E$, the *convex closure* $\overline{X}$ is the set of $v \in E$ that can be written as a nonnegative linear combination of vectors in X. This is an example of an *anti-exchange closure*, and its poset of closed sets is a meet-distributive lattice; see e.g. [33]. Recall that a lattice is meet-distributive if every co-atomic interval is Boolean. Such lattices are necessarily join-semidistributive.

A set is biclosed if it is convex and its complement is convex. In this context, such sets are usually called *biconvex*. The collection of biconvex sets need not be a lattice. For example, if $E = \{(0,0,1),(1,0,1),(-1,0,1),(0,1,1),(0,-1,1)\}$ then $E - \{(0,1,1)\}$ and $E - \{(0,-1,1)\}$ are minimal biconvex sets containing the biconvex sets $\{(-1,0,1)\}$ and $\{(1,0,1)\}$. Hence, the join of $\{(-1,0,1)\}$ and $\{(1,0,1)\}$ does not exist.

A subset X of E is *2-closed* (or *rank-2-convex closed*) if $X \cap Y$ is convex in $E \cap Y$ for any 2-dimensional linear subspace Y of $\mathbb{R}^n$. In other words, if $x, y \in X$ and $\lambda x + \mu y \in E$ for some $\lambda, \mu \in \mathbb{R}_{\geq 0}$, then $\lambda x + \mu y \in X$.

If E has no linear dependencies with three or fewer vectors, then every subset of E is biclosed. Hence, $\mathrm{Bic}(E)$ is a Boolean lattice on E if E is a generic configurations of vectors in $\mathbb{R}^n$, $n \geq 3$. On the other hand, the same configuration E of five vectors from the previous example also gives an example where $\mathrm{Bic}(E)$ is not a lattice.

We note that neither of these closure operators are necessarily ordered by single-step inclusion. However, for some highly structured vector configurations, biclosed and biconvex sets are equivalent to separation sets of the associated hyperplane arrangement.

Chapter 3

Initial results

3.1 Lattice Methods

In this section, we introduce some techniques for proving that certain posets are lattices, possibly with some additional lattice properties. In §3.1.1, examine some "local" tests for the lattice property. Some basic facts about lattice congruences of finite lattices are covered in §3.1.2. We cite Reading for these results, though most of it is probably folklore. In §3.1.3, we develop some criteria for a poset of biclosed sets to be a congruence-uniform lattice.

3.1.1 BEZ-type lemmas

Given a poset P, the statement "P is a join-semilattice" is a global statement about P in two senses:

1. The join $x \vee y$ exists only if the set of *all* elements greater than both x and y has a smallest element.

2. P is a join-semilattice if $x \vee y$ exists for *any* two elements $x, y \in P$.

Björner, Edelman, and Ziegler discovered a way to replace the second statement by a local condition, as in the following lemma. This lemma is frequently used to prove that a poset is a lattice, see e.g. [14], [47], [54], [65].

Lemma 3.1.1 ([14] Lemma 2.1) *Let P be a finite poset with $\hat{0}$ and $\hat{1}$. If $x \vee y$ exists for $x, y, z \in P$ such that x and y both cover z, then P is a lattice.*

Proof: For $x \in L$, define depth(x) to be the length of the longest chain from x to $\hat{1}$. We proceed by induction on depth.

The only element of depth 0 is $\hat{1}$. The interval $[\hat{1}, \hat{1}]$ is a one-element lattice.

Fix $d \in \mathbb{N}$. Assume that for $z \in P$ such that depth$(z) < d$, if $a, b \in [z, \hat{1}]$, then $a \vee b$ exists in P.

Now assume that $z \in P$ such that depth$(z) = d$. Let $a, b \in [z, \hat{1}]$. If $w \in P$ with $w > z$ such that $a, b \in [w, \hat{1}]$, then $a \vee b$ exists by the inductive hypothesis. Hence we may assume that z is maximal with the property that $z \leq a$ and $z \leq b$. If $z = a$ or $z = b$, then $a \vee b = \max\{a, b\}$. Suppose a and b are incomparable. Since $z < a$ and $z < b$, we may choose $x, y \in P$ for which $z \lessdot x \leq a$ and $z \lessdot y \leq b$. By assumption, the join $x \vee y$ exists. Since depth$(x) < d$ and depth$(y) < d$, the joins $a \vee (x \vee y) = a \vee y$ and $b \vee (x \vee y) = b \vee x$ both exist. Since depth$(x \vee y) < d$, the join $(a \vee y) \vee (b \vee x) = a \vee b$ exists.

Since P is a finite bounded join-semilattice, it is a lattice. ∎

In [79], Nathan Reading collected a variety of similar results, which he calls *BEZ-type lemmas*. For our purposes, we only require the following lemmas, discovered independently by the author.

Lemma 3.1.2 *Let $f : L \to L'$ be an order-preserving map between finite lattices L and L'.*

1. *Suppose $f(x \vee y) = f(x) \vee f(y)$ for $x, y, z \in L$ such that x and y both cover z. Then $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in L$.*
2. *Suppose $f(x) = f(y)$ implies $f(x \vee y) = f(x)$ for $x, y, z \in L$ such that x and y both cover z. If f preserves meets, then $f(a) = f(b)$ implies $f(a \vee b) = f(a)$ for all $a, b \in L$.*

Proof: We prove both statements by induction on depth (see proof of Lemma 3.1.1).

(1): Let $a, b \in L$. Assume $f(a' \vee b') = f(a') \vee f(b')$ whenever $a \wedge b < a' \wedge b'$. If $a \leq b$ then $f(a \vee b) = f(b) = f(a) \vee f(b)$ since f is order-preserving.

Assume a and b are incomparable, and let x and y cover $a \wedge b$ such that $x \leq a$ and $y \leq b$. Then $x \neq y$ and $f(x \vee y) = f(x) \vee f(y)$ by assumption. Since $x \leq a \wedge (x \vee y)$ holds, we have $f(a) \vee f(x \vee y) = f(a \vee x \vee y) = f(a \vee y)$ by induction. Similarly, $f(b) \vee f(x \vee y) = f(b \vee x \vee y) = f(b \vee x)$ holds. Since $x \vee y \leq (a \vee y) \wedge (b \vee x)$, we deduce

$$\begin{aligned} f(a \vee b) = f(a \vee y \vee b \vee x) = f(a \vee y) \vee f(b \vee x) &= f(a) \vee f(x \vee y) \vee f(b) \\ &= f(a) \vee f(b) \vee f(x) \vee f(y) \\ &= f(a) \vee f(b). \end{aligned}$$

(2): Assume f preserves meets. Let $a, b \in L$ such that $f(a) = f(b)$, and set $w = f(a)$. If $a \leq b$, then $f(a \vee b) = f(b) = f(a)$ holds.

Assume a and b are incomparable, and let x and y cover $a \wedge b$ such that $x \leq a$ and $y \leq b$. Since $f(a) = f(b) = w$ and f preserves meets, we have $f(a \wedge b) = w$. As f is order-preserving, this implies $f(x) = w = f(y)$. In particular, $f(x \vee y) = w$ by assumption. As before, we deduce that $f(a \vee (x \vee y)) = w$ and $f(b \vee (x \vee y)) = w$ by the induction hypothesis. Applying the induction hypothesis again, we deduce $f(a \vee b) = w$. ∎

3.1.2 Lattice congruences

The following characterization of lattice congruences is well-known.

Proposition 3.1.3 *Let Θ be an equivalence relation on a finite lattice L. If*

1. *the equivalence classes of Θ are all closed intervals of L, and*
2. *the maps $x \mapsto \min[x]$ and $x \mapsto \max[x]$ taking an element of L to the smallest (respectively, largest) element of its equivalence class are both order-preserving,*

then Θ is a lattice congruence. Conversely, every lattice congruence satisfies (1) and (2).

Proof: First assume Θ is a lattice congruence. Let $x \in L$ be given. Since L is finite, the equivalence class $[x]$ is a finite set, so $\bigwedge[x]$ and $\bigvee[x]$ exist and are equivalent to x.

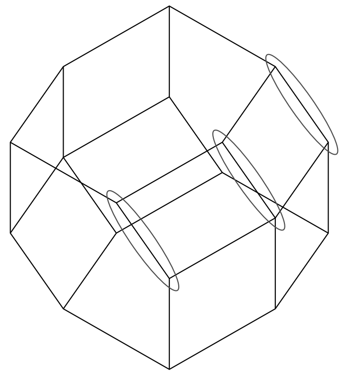


Figure 3.1: A lattice congruence on $\mathrm{Ch}(\mathcal{A}, c_0)$ of Figure 4.1.

In particular, the class $[x]$ is a subset of the interval $[\bigwedge[x], \bigvee[x]]$. If $y \in [\bigwedge[x], \bigvee[x]]$ then

$$y = y \wedge \bigvee[x] \equiv y \wedge \bigwedge[x] = \bigwedge[x] \equiv x \mod \Theta,$$

so $y \in [x]$.

We next show $x \mapsto \min[x]$ is order-preserving. Let $x, y \in L$ such that $x \leq y$. Then $x = x \wedge y \equiv x \wedge \min[y] \mod \Theta$ holds, so $\min[x] \leq x \wedge \min[y]$, which implies $\min[x] \leq \min[y]$. A similar argument shows that $x \mapsto \max[x]$ is order-preserving.

Now assume (1) and (2). Let $x, y, z \in L$ such that $x \equiv y \mod \Theta$. By (1), we deduce $(x \wedge y) \equiv x \mod \Theta$, so $\min[x] \leq x \wedge y$. By (2), $\min[x \wedge z] \leq \min[x]$ holds, so $\min[x \wedge z] \leq y$. Consequently, $\min[x \wedge z] \leq x \wedge y \wedge z \leq x \wedge z$. By (1), we deduce $(x \wedge y \wedge z) \equiv (x \wedge z) \mod \Theta$. By a parallel argument, $(x \wedge y \wedge z) \equiv (y \wedge z) \mod \Theta$. Therefore, Θ preserves meets. By a dual argument, Θ also preserves joins. Hence, it is a lattice congruence. ∎

In practice, it may be simpler to specify a lattice congruence by a pair of maps as in the following lemma.

Lemma 3.1.4 *Let L be a finite lattice, and let $\pi_\downarrow, \pi^\uparrow$ be idempotent, order-preserving maps on L such that for all $x \in L$,*

1. $\pi_\downarrow(x) \leq x \leq \pi^\uparrow(x)$,

2. $\pi_\downarrow(\pi^\uparrow(x)) = \pi_\downarrow(x)$, *and*

3. $\pi^{\uparrow}(\pi_{\downarrow}(x)) = \pi^{\uparrow}(x)$.

Then the relation $x \equiv y \mod \Theta$ *if* $\pi_{\downarrow}(x) = \pi_{\downarrow}(y)$ *is a lattice congruence with equivalence classes* $[\pi_{\downarrow}(x), \pi^{\uparrow}(x)]$ *for* $x \in L$.

Proof: It is clear that the relation $x \equiv y \mod \Theta$ is an equivalence relation. We show that the equivalence classes are intervals.

Let $x \in L$, and suppose $\pi_{\downarrow}(y) = \pi_{\downarrow(x)}$. Then

$$y \leq \pi^{\uparrow}(y) = \pi^{\uparrow}(\pi_{\downarrow}(y)) = \pi^{\uparrow}(\pi_{\downarrow}(x)) = \pi^{\uparrow}(x).$$

Hence, $y \in [\pi_{\downarrow}(x), \pi^{\uparrow}(x)]$. Conversely, if $y \in [\pi_{\downarrow}(x), \pi^{\uparrow}(x)]$, then

$$\pi_{\downarrow}(x) = \pi_{\downarrow}(\pi_{\downarrow}(x)) \leq \pi_{\downarrow}(y) \leq \pi_{\downarrow}(\pi^{\uparrow}(x)) = \pi_{\downarrow}(x),$$

so they are all equal.

The maps $x \mapsto \min[x]_{\Theta}$ and $x \mapsto \max[x]_{\Theta}$ are equal to $\pi_{\downarrow}$ and $\pi^{\uparrow}$, which are order-preserving by hypothesis. By Proposition 3.1.3, we deduce that this is a lattice congruence. ∎

Lattice congruences naturally descend to quotients, as in the following lemma.

Lemma 3.1.5 (Reading [75] Lemma 2.1) *Let* Θ *be a lattice congruence of* L *and* $[x, y]$ *an interval of* L. *The restriction of* Θ *to* $[x, y]$ *is a lattice congruence of* $[x, y]$. *Moreover, the interval* $[[x], [y]]$ *of* L/Θ *is isomorphic to* $[x, y]/\Theta$.

Proof: Since $[x, y]$ is a sublattice of L, the restriction of Θ is a lattice congruence on $[x, y]$.

If $a \in [x, y]$, then $[a] \in [[x], [y]]$ is immediate. Now let $a \in L$ such that $[x] \leq [a] \leq [y]$. Since $x \leq \max[a]$, $[a] = [x \vee \max[a]] = [x \vee \min[a]]$ holds. But as $y \geq \min[a]$, we have $y \geq x \vee \min[a]$. Hence, $x \vee \min[a] \in [x, y]$ and $[x \vee \min[a]] = [a]$.

Consequently, $[x, y]/\Theta$ and $[[x], [y]]_{L/\Theta}$ are equal as sets. To see that they have the same order structure, we observe that the above construction $x \vee \min[a]$ of an element of $[x, y]$ equivalent to a is order-preserving. ∎

Covering relations behave nicely under lattice quotients. We take advantage of this result in our work on Crosscut-simplicial lattices and the Grid-Tamari orders.

Lemma 3.1.6 (Reading [75] Proposition 2.2) *The atoms of L/Θ are in bijection with the set of elements covering $\pi^{\uparrow}(\hat{0})$ via the map $a \mapsto [a]$.*

Proof: Let $x = \pi^{\uparrow}(\hat{0})$. We verify that the above map $a \mapsto [a]$ is a well-defined, bijective map from covers of x to covers of $[x]$.

Let $a \in L$ be a cover of x. If $[b] < [a]$, then $[b \vee x] = [b]$, which implies $\pi^{\downarrow}(b) \vee x < a$. Since a covers x, this forces $[b] = [\hat{0}]$.

Assume a, b cover x such that $[a] = [b]$. Since $[a \wedge b] = [a]$, $[a] \neq [x]$, we have $x < a \wedge b \leq a$. But a covers x, so $a \leq b$. Similarly, $b \leq a$.

Assume $[a]$ covers $[x]$ and let a' be the smallest element in the class $[a]$ larger than x. If $x < b < a'$ for some $b \in L$, then $[x] < [b] < [a]$, an impossibility. Hence, a' covers x. ∎

3.1.3 Lattice properties of biclosed sets

For fixed $z \in L$, the map $x \mapsto x \wedge z$ is an order-preserving map $L \to L$ that preserves meets. Thus, Lemma 3.1.2(2) determines a local test for meet-semidistributivity. Applying this lemma to posets of biclosed sets, we get the following test for semidistributivity.

Theorem 3.1.7 *Let S be a set with a closure operator. If*

1. $\mathrm{Bic}(S)$ *is ordered by single-step inclusion, and*

2. $W \cup \overline{(X \cup Y) - W}$ *is biclosed for* $W, X, Y \in \mathrm{Bic}(S)$ *with* $W \subseteq X \cap Y$*,*

then $\mathrm{Bic}(S)$ *is a semidistributive lattice.*

Proof: If $W, X, Y \in \mathrm{Bic}(S)$ with $W \subseteq X \cap Y$, then

$$X \cup Y \subseteq W \cup \overline{(X \cup Y) - W} \subseteq \overline{X \cup Y},$$

so $X \vee Y$ and $W \cup \overline{(X \cup Y) - W}$ are equal if the latter is biclosed. Taking $W = \emptyset$, condition (2) implies $\mathrm{Bic}(S)$ is a lattice.

Since $\mathrm{Bic}(S)$ is a self-dual poset, semidistributivity follows from meet-semidistributivity. By the above discussion, it suffices to show for $W, X, Y, Z \in \mathrm{Bic}(S)$ if X and Y both cover W and $X \wedge Z = Y \wedge Z$, then $(X \vee Y) \wedge Z = X \wedge Z$.

By (1), there exists $s, t \in S$ such that $X = W \cup \{s\}$ and $Y = W \cup \{t\}$. By (2), $X \vee Y = W \cup \overline{\{s,t\}}$. If $W \wedge Z < (X \vee Y) \wedge Z$, then there exists $u \in (X \vee Y) \wedge Z$ such that $(W \wedge Z) \cup \{u\}$ is biclosed. Then u is an element of $(X \vee Y) - W$, so $u \in \overline{\{s,t\}}$. Since $W \wedge Z = X \wedge Z = Y \wedge Z$, the elements s, t are not in $W \wedge Z$ and $u \neq s,\ u \neq t$. However, this implies $\{s,t\}$ is contained in the complement of $(W \wedge Z) \cup \{u\}$, contradicting the assumption that this set is biclosed. Hence, $W \wedge Z = (X \vee Y) \wedge Z$ holds. ∎

Example 3.1.8 *The weak order on permutations may be identified with a collection of "biclosed" subsets of $\binom{[n]}{2}$, ordered by inclusion. A subset X of $\binom{[n]}{2}$ is closed if $\{i,k\}$ is in X whenever $\{i,j\}$ and $\{j,k\}$ are in X for some j with $i < j < k$. Then X is biclosed if both X and $\binom{[n]}{2} - X$ are closed. The map taking a permutation to its inversion set is an isomorphism between the weak order and the poset of biclosed subsets of $\binom{[n]}{2}$.*

More generally, the weak order on any finite Coxeter group may be identified with a poset of biclosed sets of positive roots ordered by inclusion. That these posets are ordered by single-step inclusion is well-known. Dyer proved that $W \cup \overline{(X \cup Y) - W}$ is a biclosed set whenever W, X, Y are biclosed and $W \subseteq X \cap Y$ [30]. He also proved this holds for infinite root systems if $\overline{X \cup Y}$ is finite. By Theorem 3.1.7 we may deduce that the weak order for finite Coxeter groups is a semidistributive lattice. Other proofs of semidistributivity appear in [59] and [74].

Theorem 3.1.9 *Let $(S, \prec)$ be a poset with a closure operator. Assume that*

1. $\mathrm{Bic}(S)$ *is ordered by single-step inclusion,*

2. $W \cup \overline{(X \cup Y) - W}$ *is biclosed for* $W, X, Y \in \mathrm{Bic}(S)$ *with* $W \subseteq X \cap Y$*, and*

3. *if* $x, y, z \in S$ *with* $z \in \overline{\{x,y\}} - \{x,y\}$ *then* $x \prec z$ *and* $y \prec z$.

Then $\mathrm{Bic}(S)$ *is a congruence-uniform lattice.*

Proof: By Theorem 3.1.7, we know that $\mathrm{Bic}(S)$ is a semidistributive lattice. To prove congruence-normality, we verify that $\mathrm{Bic}(S)$ admits a CN-labeling. Since $\mathrm{Bic}(S)$ is self-dual, the dual conditions will follow from (CN1)-(CN3).

By (1), we may label a covering relation $X \lessdot Y$ by the unique element in $Y - X$. These labels are partially ordered by $\prec$. The property (CN3) is immediate from this definition.

Let $W, X, Y \in \mathrm{Bic}(S)$ such that X, Y both cover W. Let $s, t \in S$ where $X = W \cup \{s\}$ and $Y = W \cup \{t\}$. By (2), $X \vee Y = W \cup \overline{\{s,t\}}$ holds, so all of the labels in $[W, X \vee Y]$ lie in $\overline{\{s,t\}}$. If C_1 is a maximal chain in $[X, X \vee Y]$, then the set $X' \in C_1$ covered by $X \vee Y$ must be of the form $(X \vee Y) - \{t\}$ as otherwise it would not be biclosed. Hence (CN1) is satisfied. Using the relation (3), (CN2) is also satisfied. ∎

Example 3.1.10 *For the closure operator on $\binom{[n]}{2}$ in Example 3.1.8, we define $\{i,j\} \preceq \{k,l\}$ if $k \leq i < j \leq l$ holds. By the discussion in Example 3.1.8, this closure operator satisfies the conditions of Theorem 3.1.9, so the weak order on permutations is a congruence-uniform lattice. This holds more generally for the weak order of any finite Coxeter group ([21, Theorem 6] or [74, Theorem 27]).*

Theorem 3.1.11 *Let S be a set with a closure operator. Assume that*

1. $\mathrm{Bic}(S)$ *is ordered by single-step inclusion,*
2. $W \cup \overline{(X \cup Y) - W}$ *is biclosed for* $W, X, Y \in \mathrm{Bic}(S)$ *with* $W \subseteq X \cap Y$*, and*
3. *for $x, y \in S$, the restriction of the closure operator to $\overline{\{x,y\}}$ is isomorphic to an order-convex closure on a chain.*

Then $\mathrm{Bic}(S)$ *is a polygonal lattice.*

Proof: From (2), we know that $\mathrm{Bic}(S)$ is a lattice. It remains to show polygonality.

Let $X, Y, W \in L$ such that X and Y both cover W. By (1), there exists $X - W = \{x\}$ and $Y - W = \{y\}$ for some $x, y \in S$. By (2), the join $X \vee Y$ is equal to $W \cup \overline{\{x,y\}}$. Since W is co-closed, W and $\overline{\{x,y\}}$ are disjoint.

By (3), there is a total order $\prec$ on $\overline{\{x,y\}}$ such that the closed subsets of $\overline{\{x,y\}}$ are the order-convex subsets of $\prec$. If $Z \subseteq \overline{\{x,y\}}$ such that $W \cup Z$ is biclosed, then Z is biclosed in $\overline{\{x,y\}}$. In particular, since X and Y are biclosed, we may assume without loss of generality that x and y are the bottom and top elements of $\prec$, respectively. As there is a unique maximal chain of order ideals from $\{x\}$ to $\overline{\{x,y\}}$ with respect to $\prec$, there is a unique maximal chain in $[X, X \vee Y]$. Since $\mathrm{Bic}(S)$ is ordered by single-step inclusion, $W \cup I$ is biclosed for any order ideal I of $(\overline{\{x,y\}}, \prec)$. By a similar argument, there is a unique maximal chain in $[Y, X \vee Y]$ consisting of sets $W \cup F$ where F is an order filter of $(\overline{\{x,y\}}, \prec)$.

As $\mathrm{Bic}(S)$ is a self-dual lattice, we are done. ∎

3.2 Hyperplane Arrangement Methods

We give some basic results on chambers and galleries of real hyperplane arrangements in §3.2.1 and §3.2.2. We use these results in later sections to study collections of biclosed sets in hyperplane arrangements.

Biclosed, biconvex, and separable sets are often defined for vector configurations rather than for hyperplane arrangements. In §3.2.3, we give a dictionary that translates between these two settings.

3.2.1 Chambers

Proposition 3.2.1 (see [32]) *Let $\mathcal{A}$ be an arrangement with a fundamental chamber c_0.*

1. *If $X \in L(\mathcal{A})$, $x \in \mathcal{L}(\mathcal{A})$ with $x^{-1}(0) = \mathcal{A}_X$, then the set of chambers incident to x forms an interval $[x \circ c_0, x \circ (-c_0)]$ of $\mathrm{Ch}(\mathcal{A}, c_0)$ isomorphic to $\mathrm{Ch}(\mathcal{A}_X, (c_0)_X)$.*

2. *$\mathrm{Ch}(\mathcal{A}, c_0)$ is a bounded, graded poset with rank function $c \mapsto |S(c)|$.*

3. *For $c, c' \in \mathrm{Ch}(\mathcal{A})$, if $\mathcal{W}(c) \subseteq S(c, c')$ then $c' = -c$.*

4. *For $c \in \mathrm{Ch}(\mathcal{A})$, $X \in L(\mathcal{A})$, if c is incident to X, then there exists a chamber c' such that $S(c, c') = \mathcal{A}_X$.*

Proof: (1) Since $S(x \circ c_0, x \circ (-c_0)) = \mathcal{A}_X$ and $S(x \circ c_0) \cap \mathcal{A}_X = \emptyset$, the set of chambers incident to x forms an interval of $\mathrm{Ch}(\mathcal{A}, c_0)$ with bottom element $x \circ c_0$ and top element $x \circ (-c_0)$.

The restriction map $[x \circ c_0, x \circ (-c_0)] \to \mathrm{Ch}(\mathcal{A}_X, (c_0)_X)$ is injective since chambers incident to x may only differ by hyperplanes in $\mathcal{A}_X$. Given a chamber c of $\mathcal{A}_X$, there exists a chamber $\hat{c}$ of $\mathcal{A}$ with $\hat{c}(H) = c(H)$ for $H \in \mathcal{A}_X$. Since $x \circ \hat{c}$ is a chamber incident to x, the restriction map is surjective.

(2) The chamber poset $\mathrm{Ch}(\mathcal{A}, c_0)$ has lower bound c_0 and upper bound $-c_0$. Let $c, c' \in \mathrm{Ch}(\mathcal{A}, c_0)$, $c < c'$ and assume $|S(c, c')| \geq 2$. We show that c' does not cover c by induction on the rank of $\mathcal{A}$. If $H \in S(c, c')$, then by (L3) (see §2.3.1) there exists $x \in \mathcal{L}(\mathcal{A})$ such that $x(H) = 0$ and $x(H') = (c \circ c')(H)$ if $H \in \mathcal{A} - S(c, c')$. Then $c \leq x \circ c_0 < x \circ (-c_0) \leq c'$ holds. If $x \neq \mathbf{0}$ and $x^{-1}(0) \neq \{H\}$, then c' does not cover c by part (1) and the induction hypothesis. If $x^{-1}(0) = \{H\}$, then the assumption $|S(c, c')| \geq 2$ implies $c < x \circ c_0$ or $x \circ (-c_0) < c'$. If $x = \mathbf{0}$ then $c = c_0$, $c' = -c_0$ and since the rank of $\mathcal{A}$ is at least 2, there exists a chamber d satisfying $c < d < c'$.

(3) By (L1), opposite chambers have the same set of walls. By part (2) there exists a chain $c' = c_1 < \cdots < c_t = -c$ of $\mathrm{Ch}(\mathcal{A}, c)$ such that $|S(c_i, c_{i+1})| = 1$ for all i. If $t > 1$, the hyperplane H separating c_{t-1} and c_t is a wall of c but H does not separate c and c'.

(4) Let c be a chamber incident to an intersection subspace X. Let $x \in \mathcal{L}(\mathcal{A})$ such that $x^{-1}(0) = X$ such that $c = x \circ c$. Let $c' = x \circ (-c)$. Then c' is incident to x and c and c' are separated by the hyperplanes in $\mathcal{A}_X$. ∎

Combining parts 1 and 3 of Proposition 3.2.1, we deduce the following corollary.

Corollary 3.2.2 *Let $\mathcal{A}$ be an arrangement with fundamental chamber c_0 and covector x. Then the join*

$$\bigvee \{c \in \mathrm{Ch}(\mathcal{A}, c_0) |\ c \text{ covers } x \circ c_0\}$$

exists and is equal to $x \circ (-c_0)$.

This corollary with Lemma 3.1.1 implies the following.

Corollary 3.2.3 *Let $\mathcal{A}$ be an arrangement with fundamental chamber c_0. If c is incident to $H \cap H'$ for all chambers c and hyperplanes $H, H' \in U(c)$, then $\mathrm{Ch}(\mathcal{A}, c_0)$ is a lattice.*

An arrangement that satisfies the hypothesis of Corollary 3.2.3 is called *bineighborly*. In Theorem 5.5.1, we prove that chamber posets are semidistributive lattices if and only if they are bineighborly.

The chambers of an arrangement completely determine its oriented matroid structure. Mandel proved that a sign vector $x \in \{0,+,-\}^E$ is a covector of an oriented matroid with tope set Ch if and only if $x \circ c$ is in Ch for all $c \in \mathrm{Ch}$ (see [15, Theorem 4.2.13]). We give a variation of this result.

Theorem 3.2.4 *A chamber $c \in \mathrm{Ch}(\mathcal{A})$ is incident to $X \in L(\mathcal{A})$ if and only if for $Y \in L(\mathcal{A}_X)$ there exists a chamber c' such that $S(c,c') = \mathcal{A}_Y$ whenever Y is incident to c_X.*

Proof: Assume c is incident to X, and let $x \in \mathcal{L}(\mathcal{A})$ such that $x^{-1}(0) = \mathcal{A}_X,\ x \leq c$. The chamber $x \circ (-c)$ satisfies $S(c, x \circ (-c)) = \mathcal{A}_X$. Since the interval $[c, x \circ (-c)]$ is isomorphic to $\mathrm{Ch}(\mathcal{A}_X, c_X)$, any wall of c_X is a wall of c.

Now assume for $Y \in L(\mathcal{A}_X)$ there exists a chamber c' such that $S(c,c') = \mathcal{A}_Y$ whenever c_X is incident to Y. We prove that c is incident to X by induction on the codimension of X. By the inductive hypothesis, c is incident to Y if $Y \in L(\mathcal{A}_X)$ and c_X is incident to Y.

Let c' be the chamber with $S(c,c') = \mathcal{A}_X$ and let $H \in \mathcal{W}(c') \cap \mathcal{A}_X$. If $H = X$, then we are done by property (L3) (see §2.3.1). Thus we assume that the codimension of X is at least 2. Since H is a wall of c'_X, it is a wall of c_X. By the inductive hypothesis, the chamber c is incident to H.

Let $Y \in L((\mathcal{A}^H)_X),\ Y \neq H$. If $Y \neq X$, there exists a chamber $d \in \mathrm{Ch}(\mathcal{A})$ such that $S(c,d) = \mathcal{A}_Y$ and d is incident to H. Hence, the chamber d^H satisfies $S(c^H, d^H) = (\mathcal{A}^H)_Y$. If $Y = X$, then $S(c^H, (c')^H) = \mathcal{A}_X$. By the inductive hypothesis c^H is incident to X. Hence, c is incident to X. ∎

3.2.2 Galleries

For $c, c' \in \mathcal{T}(\mathcal{A})$, the set of reduced galleries from c to c' forms a graph $\mathrm{Gal}(c,c')$ where galleries r and r' are adjacent if $|L_2(r,r')| = 1$. Alternatively, one may define adjacency of galleries by "flipping" about a codimension 2 face. The equivalence of these two definitions is shown in the following proposition.

Proposition 3.2.5 *If r is a reduced gallery from c to c' and X a codimension 2 intersection subspace, then there exists a reduced gallery r' such that $L_2(r,r') = \{X\}$ if and only if r is incident to X.*

Proof: Assume r is incident to X. Let $x \in \mathcal{L}(\mathcal{A})$ with $x^{-1}(0) = \mathcal{A}_X$ such that $x \circ c$ and $x \circ (-c)$ are chambers in r. By Proposition 3.2.1(1), the interval $[x \circ c, x \circ (-c)]$ of $\mathrm{Ch}(\mathcal{A}, c)$ is isomorphic to $\mathrm{Ch}(\mathcal{A}_X, c_X)$. Since $\mathcal{A}_X$ is of rank 2, the interval $[x \circ c, x \circ (-c)]$ has two maximal chains.

Let r' be the symmetric difference of r with the open interval $(x \circ c, x \circ (-c))$. Then r' is a reduced gallery from c to c' such that $X \in L_2(r,r')$. Let $Y \in L_2(\mathcal{A})$, $Y \neq X$. Since every hyperplane in $S(x \circ c, x \circ (-c))$ contains X, the localized arrangement $\mathcal{A}_Y$ contains at most one hyperplane of $S(x \circ c, x \circ (-c))$. Hence, $d_Y = (x \circ c)_Y$ or $d_Y = x \circ (-c)_Y$ for $d \in [x \circ c, x \circ (-c)]$, so $L_2(r,r') = \{X\}$.

Now assume r' is a reduced gallery from c to c' such that $L_2(r,r') = \{X\}$. Let d be the largest chamber common to r and r' for which $r_{\leq d} = r'_{\leq d}$. Let H and H' be the upper walls of d crossed by r and r'. Since r and r' are separated by $H \cap H'$, both H and H' contain X. Let $x \in \{0,+,-\}^{\mathcal{A}}$ such that $x(H'') = 0$ if $H'' \in \mathcal{A}_X$ and $x(H'') = d(H'')$ otherwise.

If $x \circ (-c)$ is not a chamber in r, then there exists a hyperplane $H'' \in \mathcal{A}$ not containing X such that r crosses H'' before H' but after H. Then r and r' are separated by $H' \cap H''$, an impossibility. Hence, $x \circ (-c)$ is a chamber. By Theorem 3.2.4, we conclude that x is a covector of $\mathcal{A}$ and r is incident to x. ∎

A fundamental fact about reduced galleries is that $\mathrm{Gal}(c,c')$ is a *connected* graph for any chambers c, c'. In §4.1, we prove that $\mathrm{Gal}(c,c')$ is exhibits some higher connectivity when $\mathcal{A}$ is bineighborly.

We say a permutation $\pi : H_1, H_2, \ldots$ of $\mathcal{A}$ is *admissible* if for each codimension 2 subspace X, there exists a gallery of $\mathcal{A}_X$ from $(c_0)_X$ to $-(c_0)_X$ crossing the hyperplanes in the order defined by π. If $\mathcal{A}$ is a reflection arrangement, an admissible permutation is called a *reflection order*. If $c_0, c_1, \ldots$ is a gallery of $\mathcal{A}$ then $H_1, H_2, \ldots$ is an admissible

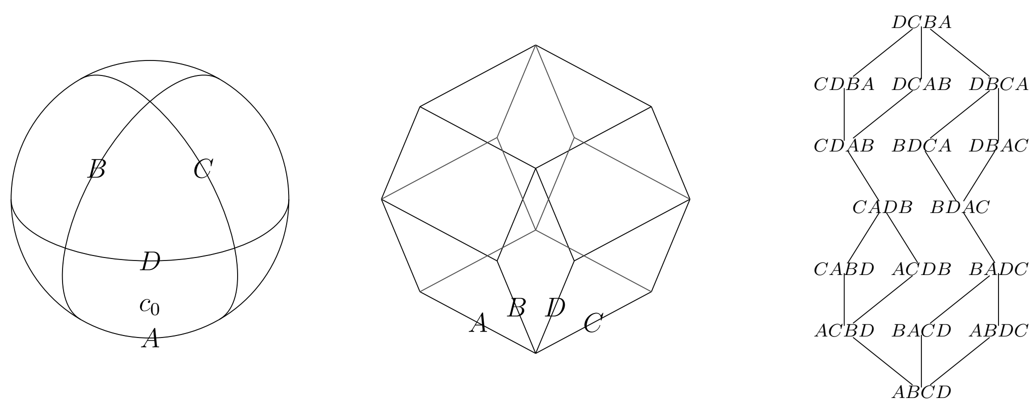


Figure 3.2: (left) An arrangement of four planes in $\mathbb{R}^3$. (center) The poset of chambers. (right) The graph of reduced galleries from c_0 to $-c_0$. The galleries $BACD$ and $CDBA$ are separated by four codimension 2 subspaces, but the shortest path between them in the gallery graph has length six.

permutation of $\mathcal{A}$ where $S(c_{i-1}, c_i) = \{H_i\}$. The following proposition gives a partial converse.

Proposition 3.2.6 *Let $\mathcal{A}$ be an arrangement with fundamental chamber c_0 such that every biclosed set is the separation set of some chamber. If $\pi : H_1, H_2, \ldots, H_N$ is an admissible permutation of $\mathcal{A}$ then there exists a gallery $c_0, c_1, \ldots, c_N = -c_0$ such that $S(c_{i-1}, c_i) = \{H_i\}$ for all i.*

Proof: For $0 \le j \le N$, let $I_j = \{H_1, \ldots, H_j\}$. Let $X \in L_2(\mathcal{A})$, and suppose

$$\mathcal{A}_X = \{H_{p_1}, \ldots, H_{p_t}\}, \qquad p_1 < \cdots < p_t.$$

Since π is admissible, there exists a chamber c of $\mathcal{A}_X$ such that $S((c_0)_X, c) = \{H_{p_k} \mid p_k \le i\} = I_j \cap \mathcal{A}_X$. Hence, I_j is biclosed, and there exists a chamber c_j such that $S(c_j) = I_j$. The chain $c_0 < c_1 < \cdots < c_N$ is a reduced gallery of $\mathcal{A}$ inducing π. ∎

For the generic arrangement of four planes in $\mathbb{R}^3$ shown in Figure 3.2, all 16 subsets of hyperplanes are biclosed, but there are only 14 chambers. This example shows the necessity of the conditions in Proposition 3.2.6, as all 24 permutations of the hyperplanes are admissible, but only 16 come from reduced galleries.

Admissible permutations are easier to flip than reduced galleries, as described in the following lemma.

Lemma 3.2.7 *Let $H_1, H_2, \ldots, H_N$ be an admissible permutation of $\mathcal{A}$. Suppose the set of hyperplanes containing some codimension 2 subspace X is a contiguous subsequence $H_i, H_{i+1}, \ldots, H_j$. Then the permutation*

$$H_1, \ldots, H_{i-1}, H_j, \ldots, H_i, H_{j+1}, \ldots, H_N$$

obtained by flipping the subsequence $H_i, \ldots, H_j$ is an admissible permutation.

Proof: Let $\pi : H_1, \ldots, H_N$ be the original permutation and let π' be the flip. If Y is some codimension 2 subspace not contained in at least two hyperplanes in $H_i, \ldots, H_j$, then the restrictions of π and π' to $\mathcal{A}_X$ are the same. If Y does contain at least two hyperplanes in the sequence, then it equals X. Since $H_i, \ldots, H_j$ is an admissible permutation of $\mathcal{A}_X$, so is the reverse $H_j, \ldots, H_i$. ∎

Once again, the arrangement of Figure 3.2 gives an example of a gallery that cannot be flipped as in Lemma 3.2.7. For instance, the gallery from c_0 to $-c_0$ crossing hyperplanes B, D, C, A in that order is not incident to $C \cap D$, so it does not admit a flip to B, C, D, A. However, the permutation B, C, D, A is an admissible permutation of $\mathcal{A}$.

3.2.3 Biclosed, Biconvex, and Separable sets

Definition 3.2.8 *Let $\mathcal{A}$ be an arrangement with fundamental chamber c_0, and let $I \subseteq \mathcal{A}$.*

- *I is a* separable set *if there exists a chamber $c \in \mathrm{Ch}(\mathcal{A})$ with $I = S(c)$.*
- *I is* convex *if for $H \in \mathcal{A} - I$ there exists a chamber c with $H \in S(c)$ and $S(c) \subseteq \mathcal{A} - I$.*
- *I is* 2-closed *if for $X \in L_2(\mathcal{A})$ the set $I \cap \mathcal{A}_X$ is convex in $\mathcal{A}_X$ with fundamental chamber $(c_0)_X$.*

- *The* convex closure *(*2-closure*) of I is the smallest convex (2-closed) set containing I.*
- *I is* biconvex *(*biclosed*) if I and $A \setminus I$ are both convex (2-closed).*

Since $\mathcal{A} - S(c) = S(-c)$ holds for $c \in \mathrm{Ch}(\mathcal{A})$, separable sets are biconvex. If $\mathcal{A}'$ is a subarrangement of $\mathcal{A}$ and I is convex in $\mathcal{A}$, then $I \cap \mathcal{A}'$ is convex in $\mathcal{A}'$. We deduce the following proposition.

Proposition 3.2.9 (see [57], Section 2.1) *If I is a separable set, then I is biconvex. If I is convex, then I is 2-closed. In particular, separable sets are biclosed.*

If $\mathcal{A}$ is of rank 2, then separable, biconvex, and biclosed sets coincide. Unlike the biconvex property, we prove that the biclosed property does not depend on the choice of a fundamental chamber.

Lemma 3.2.10 *Let $\mathcal{A}$ be a hyperplane arrangement with chambers $c_0, c \in \mathrm{Ch}(\mathcal{A})$, and let $I \subseteq \mathcal{A}$. The set I is biclosed with respect to c_0 if and only if $I \bigtriangleup S(c_0, c)$ is biclosed with respect to c.*

Proof: Assume I is biclosed with respect to c_0 and let $X \in L_2(\mathcal{A})$. Since $I \cap \mathcal{A}_X$ is biclosed in $\mathcal{A}_X$ with respect to $(c_0)_X$, there exists a chamber $d \in \mathrm{Ch}(\mathcal{A}_X)$ such that $S((c_0)_X, d) = I \cap \mathcal{A}_X$. We have

$$S(c_X, d) = S((c_0)_X, d) \bigtriangleup S((c_0)_X, c_X) = (I \bigtriangleup S(c_0, c)) \cap \mathcal{A}_X.$$

Hence, $I \bigtriangleup S(c_0, c)$ is biclosed with respect to c. ∎

Lemma 3.2.11 *If $c, d \in \mathrm{Ch}(\mathcal{A}, c_0)$ such that $c \leq d$, then $S(c, d)$ is convex.*

Proof: Since $c \leq d$, the arrangement $\mathcal{A}$ has a partition into three disjoint subsets $S(c_0, c), S(c, d)$, and $S(d, -c_0)$. Since $S(c_0, c) = S(c)$ and $S(d, -c_0) = S(-d)$, every hyperplane in $\mathcal{A} - S(c, d)$ is either in $S(c)$ or $S(d)$. ∎

Definition 3.2.8 has a polar dual analogue. Given an oriented hyperplane H in a real vector space V, let $v_H \in V^*$ be the unit vector with $v_H^{-1}(\mathbb{R}_{>0}) = H^+$. The association $H \mapsto v_H$ defines a correspondence between (oriented) real central hyperplane arrangements and configurations of unit vectors. Separable, convex, and 2-closed subsets of an acyclic vector configuration are usually defined as in the following proposition.

Proposition 3.2.12 *Let I be a subset of hyperplanes of an arrangement $\mathcal{A}$ with fundamental chamber c_0. Assume $c_0(H) = +$ for all $H \in \mathcal{A}$.*

1. *I is a separable set if and only if there does not exist a circuit $v \in \{0, +, -\}^{\mathcal{A}}$ such that $v^{-1}(+) \subseteq I$ and $v^{-1}(-) \subseteq \mathcal{A} - I$.*

2. *I is convex if and only if there does not exist a circuit $v \in \{0, +, -\}^{\mathcal{A}}$ such that $v^{-1}(+) \subseteq I,\ v^{-1}(-) \subseteq \mathcal{A} - I$ and $|v^{-1}(-)| = 1$.*

3. *I is 2-closed if and only if there does not exist a circuit $v \in \{0, +, -\}^{\mathcal{A}}$ such that $v^{-1}(+) \subseteq I,\ v^{-1}(-) \subseteq \mathcal{A} - I,\ |v^{-1}(-)| = 1$, and $|v^{-1}(+)| = 2$.*

Proof: (1) For $c \in \{+, -\}^{\mathcal{A}}$, the intersection $\bigcap_{H \in \mathcal{A}} H^{c(H)}$ is nonempty if and only if $\bigcap_{H \in \mathcal{A}'} H^{c(H)}$ is nonempty for all subarrangements $\mathcal{A}'$ of $\mathcal{A}$. Hence, a sign vector c is a chamber if and only if there does not exist a circuit $v \in \{0, +, -\}^{\mathcal{A}}$ such that $c|_{v^{-1}(\{+,-\})}$ equals $v|_{v^{-1}(\{+,-\})}$.

(2) The set I is convex in $\mathcal{A}$ if and only if I is convex in $I \cup \{H\}$ for all $H \in \mathcal{A} - I$. For $H \in \mathcal{A}$, I is convex in $I \cup \{H\}$ if and only if there exists a chamber c of $I \cup \{H\}$ such that $S(c_0|_{I \cup \{H\}}, c) = \{H\}$, which holds if and only if I is separable in $I \cup \{H\}$. The statement follows from part (1).

(3) The set I is 2-closed in $\mathcal{A}$ if and only if $I \cap \mathcal{A}_X$ is convex in $\mathcal{A}_X$ for $X \in L_2(\mathcal{A})$. By part (2), this holds if and only if there does not exist a circuit v of $\mathcal{A}_X$ such that $v^{-1}(+) \subseteq I,\ v^{-1}(-) \subseteq \mathcal{A} - I,\ |v^{-1}(-) = 1|$ for $X \in L_2(\mathcal{A})$. But a circuit of $\mathcal{A}$ has three elements if and only if it is a circuit of $\mathcal{A}_X$ for some $X \in L_2(\mathcal{A})$. ∎

3.2.4 Set-valued metrics on graphs

Our notation for set-valued metrics follows [81, §3]. For this section, we fix a connected graph G and a set T. A *set-valued metric* $\delta : G \times G \to T$ is a function for which

(M1) $\delta(u, v) = \delta(v, u)$,

(M2) u is adjacent to v implies $|\delta(u,v)| = 1$, and

(M3) $\delta(u,v) = \delta(u,w) \bigtriangleup \delta(w,v)$,

for vertices $u, v, w \in G$. Here, $X \bigtriangleup Y$ is the symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ for sets X, Y. Observe that (M3) implies $\delta(u,u) = \emptyset$ for all vertices u. We denote a graph with set-valued metric by the triple (G, T, δ).

If u, v are vertices of G, a *path* from u to v is a sequence of vertices $(u_0, \ldots, u_l)$ such that $u_0 = u,\ u_l = v$ and u_{i-1} is adjacent to u_i for all i. The *length* of a path $(u_0, \ldots, u_l)$ is l. Given vertices u, v, a *geodesic* is a path from u to v of minimum length. Let $d_G : G \times G \to \mathbb{Z}$ be the *distance function*; that is $d_G(u,v)$ is the length of any geodesic from u to v. The *diameter* of G is the maximum value of $d_G(u,v)$ for vertices u, v.

The axioms for a set-valued metric resemble properties of d_G: For vertices u, v, w of G,

- $d_G(u,v) = d_G(v,u)$,
- u is adjacent to v implies $d_G(u,v) = 1$, and
- $d_G(u,v) \leq d_G(u,w) + d_G(w,v)$.

The functions $S : \mathrm{Ch}(\mathcal{A}) \times \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ and $L_2(\cdot,\cdot) : \mathrm{Gal} \times \mathrm{Gal} \to L_2(\mathcal{A})$ are both examples of set-valued metrics. The separation function $S(\cdot,\cdot)$ has the additional property that for $c, c' \in \mathrm{Ch}(\mathcal{A})$ there exists a geodesic from c to c' in the chamber graph of length $|S(c,c')|$. However, this property does not hold for L_2; see Figure 3.2. In general, we get the following inequality.

Lemma 3.2.13 ([81], Proposition 3.5) *For $u, v \in G$, $|\delta(u,v)| \leq d_G(u,v)$.*

Proof: Let $l > 0$ and assume the lemma holds if $d_G(u,v) < l$. Let $u, v \in G$ such that $d_G(u,v) = l$ and let $(u_0, \ldots, u_l)$ be a geodesic from u to v. Then $(u_0, \ldots, u_{l-1})$ is a geodesic from u to u_{l-1}, so $|\delta(u, u_{l-1})| \leq d_G(u, u_{l-1})$. Since $\delta(u,v) = \delta(u, u_{l-1}) \bigtriangleup \delta(u_{l-1}, v)$ and $|\delta(u_{l-1}, v)| = 1$, we deduce $|\delta(u,v)| \leq d_G(u, u_{l-1}) + 1 = d_G(u,v)$. The result follows by induction. ∎

A vertex u is *δ-accessible* (or simply *accessible*) if there exists a geodesic between u and v of length $|\delta(u,v)|$ for all vertices $v \in G$. An *equivariant involution* is an involution $v \mapsto -v$ on G for which $\delta(v, -v) = T$. If G has an equivariant involution, then its diameter is at least $|T|$ by Lemma 3.2.13. If $v \mapsto -v$ is an equivariant involution, then property (M3) implies $\delta(u, v) = T - \delta(u, -v) = \delta(-u, -v)$ for all $u, v \in G$. Hence, $d_G(u, v) = d_G(-u, -v)$ for all vertices u, v. In particular, if u is accessible, then so is $-u$.

Proposition 3.2.14 ([81], Proposition 3.12) *Let G be a graph with a set-valued metric $\delta : G \times G \to T$ such that (G, δ) admits an equivariant involution. If G contains an accessible vertex, then the diameter of G is equal to $|T|$.*

Proof: We must show $d_G(v, w) \leq |T|$ for all $v, w \in G$.

Let u, v, w be vertices of G, and assume u is an accessible vertex. If $|\delta(u, v)| + |\delta(u, w)| \leq |T|$, then by accessibility, there exists a path from v to w passing through u of length less than $|T|$. Hence, $d_G(v, w) \leq |T|$ in this case. If $|\delta(u, v)| + |\delta(u, w)| > |T|$, then

$$|\delta(-u, v)| + |\delta(-u, w)| = 2|T| - |\delta(u, v)| - |\delta(u, w)| \leq |T|.$$

By accessibility of $-u$, we again deduce $d_G(v, w) \leq |T|$. ∎

3.3 Poset Topology Methods

In [12, §10], Björner recorded a handful of results used to compute the homotopy type of posets. In this section, we show how to derive these results from one simple lemma:

Lemma 3.3.1 ([99] Proposition 6.1) *For $x \in P$, if $P_{<x}$ or $P_{>x}$ is contractible, then P is homotopy equivalent to $P - x$.*

3.3.1 Proof of Lemma 3.3.1

If the link or deletion of a vertex in a simplicial complex Δ is contractible, then Δ is homotopy equivalent to a smaller complex as in the following lemma. This lemma is a consequence of the Carrier Lemma (see [12] Lemma 10.1 for the Carrier Lemma).

Lemma 3.3.2 *Let Δ be a simplicial complex containing a vertex v.*

1. *If* $\mathrm{lk}_\Delta(v)$ *is contractible, then* Δ *is homotopy equivalent to* $\mathrm{dl}_\Delta(v)$.

2. *If* $\mathrm{dl}_\Delta(v)$ *is contractible, then* Δ *is homotopy equivalent to the suspension of* $\mathrm{lk}_\Delta(v)$.

Proof: (1) Assume $\mathrm{lk}_\Delta(v)$ is contractible. There are contractible carriers $C_1 : \Delta \to \Delta,\ C_{12} : \Delta \to \mathrm{dl}_\Delta(v)$ where

$$C_1(F) = \begin{cases} F & \text{if } v \notin F \\ \mathrm{st}_\Delta(v) & \text{if } v \in F \end{cases} \quad (F \in \Delta),$$

$$C_{12}(F) = \begin{cases} F & \text{if } v \notin F \\ \mathrm{lk}_\Delta(v) & \text{if } v \in F \end{cases} \quad (F \in \Delta).$$

Let $f : \|\mathrm{dl}_\Delta(v)\| \hookrightarrow \|\Delta\|$ be the inclusion, and let $g : \|\Delta\| \to \|\mathrm{dl}_\Delta(v)\|$ be a continuous function carried by C_{12}. Since $g \circ f$ and $\mathrm{id}_{\mathrm{dl}_\Delta(v)}$ are both carried by the identity on $\mathrm{dl}_\Delta(v)$, they are homotopic. Similarly, $f \circ g$ and id_Δ are homotopic since both are carried by C_1.

(2) Assume $\mathrm{dl}_\Delta(v)$ is contractible. Let $\{v', v''\}$ be a discrete set disjoint from Δ and let $\Delta' = \{v', v''\} * \mathrm{lk}_\Delta(v)$. Define contractible carriers C_1, C_2, C_{12}, C_{21} where

$$\begin{aligned}
C_1 :& \Delta \to \|\Delta\| \\
& F \mapsto \begin{cases} F & \text{if } F \in \mathrm{lk}_\Delta(v) \\ \mathrm{st}_\Delta(v) & \text{if } v \in F \\ \mathrm{dl}_\Delta(v) & \text{if } F \in \mathrm{dl}_\Delta(v) - \mathrm{lk}_\Delta(v) \end{cases} \qquad (F \in \Delta) \\
C_2 :& \Delta' \to \|\Delta'\| \\
& F \mapsto \begin{cases} F & \text{if } F \in \mathrm{lk}_\Delta(v) \\ \mathrm{st}_{\Delta'}(v') & \text{if } v' \in F \\ \mathrm{st}_{\Delta'}(v'') & \text{if } v'' \in F \end{cases} \qquad (F \in \Delta') \\
C_{12} :& \Delta \to \|\Delta'\| \\
& F \mapsto \begin{cases} F & \text{if } F \in \mathrm{lk}_\Delta(v) \\ \mathrm{st}_{\Delta'}(v') & \text{if } v \in F \\ \mathrm{st}_{\Delta'}(v'') & \text{if } F \in \mathrm{dl}_\Delta(v) - \mathrm{lk}_\Delta(v) \end{cases} \qquad (F \in \Delta) \\
C_{21} :& \Delta' \to \|\Delta\| \\
& F \mapsto \begin{cases} F & \text{if } F \in \mathrm{lk}_\Delta(v) \\ \mathrm{st}_\Delta(v) & \text{if } v' \in F \\ \mathrm{dl}_\Delta(v) & \text{if } v'' \in F \end{cases} \qquad (F \in \Delta')
\end{aligned}$$

Let $f : \|\Delta\| \to \|\Delta'\|$ and $g : \|\Delta'\| \to \|\Delta\|$ be carried by C_{12} and C_{21}, respectively. Then $g \circ f$ and id_Δ are both carried by C_1, and $f \circ g$ and $\mathrm{id}_{\Delta'}$ are carried by C_2. ∎

Lemma 3.3.1 is an immediate consequence of Lemma 3.3.2(1).

3.3.2 Quillen's Fiber Lemma

Quillen's Fiber Lemma is a powerful tool in poset topology. Although this is not the most general form of Quillen's lemma, it follows easily from Lemma 3.3.1.

Lemma 3.3.3 *Let $f : P \to Q$ be an order-preserving map of posets. If $f^{-1}(Q_{\geq x})$ is contractible for all $x \in Q$, then f induces a homotopy equivalence $P \simeq Q$.*

Proof: Suppose all of the fibers $f^{-1}(Q_{\geq x})$ are contractible. Let $q_1, \ldots, q_N$ be a linear extension of Q. We define an ordering on the disjoint union $P \sqcup Q$ extending the orders on P and Q such that if $x \in P$ and $y \in Q$ then $y \leq x$ if $y \leq f(x)$. For each i, we have

$$(P \cup \{q_1, \ldots, q_{i-1}, q_i\})_{>q_i} = \{x \in P : \ q_i \leq f(x)\} = f^{-1}(Q_{\geq q_i}),$$

where the last equality holds since f is order-preserving. Since the fibers are contractible, we deduce that $P \cup \{q_1, \ldots, q_i\} \simeq P \cup \{q_1, \ldots, q_{i-1}\}$ for all i by Lemma 3.3.1. Hence, $P \simeq P \sqcup Q$.

Now let $p_1, \ldots, p_M$ be a linear extension of P. For each i, we have

$$((P \sqcup Q) - \{p_1, \ldots, p_{i-1}\})_{<p_i} = \{y \in Q : \ y \leq f(p_i)\}.$$

The latter subposet is contractible since it contains a cone point $f(p_i)$. Then $(P \sqcup Q) - \{p_1, \ldots, p_{i-1}\}$ is homotopy equivalent to $(P \sqcup Q) - \{p_1, \ldots, p_i\}$ for all i. Hence, $P \sqcup Q \simeq Q$. ∎

For our purposes, a natural application of the Fiber Lemma is to lattice quotient maps. This result is immediate from the fiber description of lattice quotients in Proposition 3.1.3.

Lemma 3.3.4 *Let P, Q be lattice, and let f be a lattice quotient map $f : P \to Q$. If $f^{-1}(\hat{0}_Q) = \{\hat{0}_P\}$ and $f^{-1}(\hat{1}_Q) = \{\hat{1}_P\}$ then f induces a homotopy equivalence $P \simeq Q$.*

3.3.3 Rambau's Suspension Lemma

Rambau introduced the Suspension Lemma in [72] to compute the homotopy type of the Higher Bruhat order $\mathrm{HB}(n, d)$. The lemma was also used to compute the homotopy type of (both kinds of) Higher Stasheff-Tamari orders [34]. In §7, we apply this result to gallery posets. Figure 3.3 shows an application of the Suspension Lemma to the weak order on $\mathfrak{S}_4$.

Lemma 3.3.5 *For any bounded poset P, $\overline{P \times 2} \simeq \mathrm{susp}(\overline{P})$.*

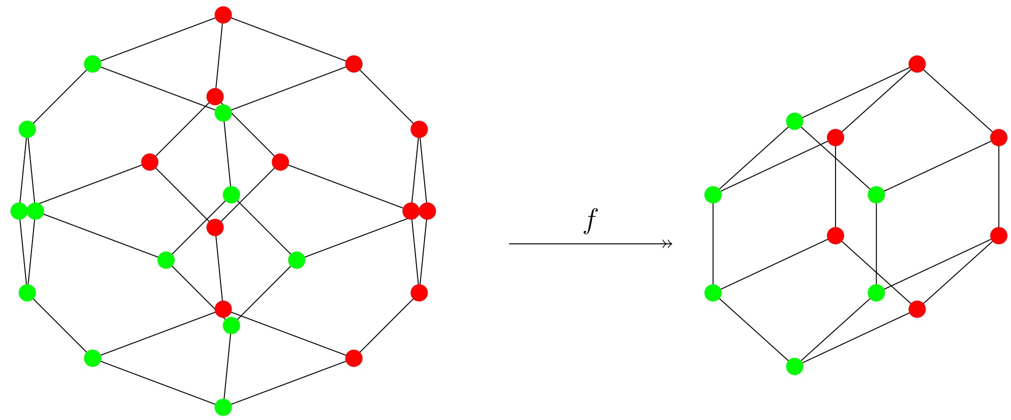


Figure 3.3: The suspension lemma applies to this map $\tilde{f} : \mathfrak{S}_4 \to (\mathfrak{S}_3 \times 2)$.

Proof: Let $Q = \overline{P} \sqcup \{a, b\}$ where $x \leq y$ whenever $x \in \overline{P}$, $y \in \{a, b\}$, and a and b are incomparable. It is clear that $\operatorname{susp} \Delta(\overline{P})$ is isomorphic to $\Delta(Q)$ are simplicial complexes.

Let $f : \overline{P \times 2} \to Q$ be the order-preserving map where

$$f(p, \epsilon) = \begin{cases} p \text{ if } p \neq \hat{1}_P, \ \epsilon = 0 \\ a \text{ if } p = \hat{1}_P, \ \epsilon = 0 \\ b \text{ if } \epsilon = 1 \end{cases} .$$

Let $q \in Q$. Then

$$f^{-1}(Q_{\geq q}) = \begin{cases} \overline{P \times 2}_{\geq (q,0)} \text{ if } q \neq a, \ q \neq b \\ \{(\hat{1}_P, 0)\} \text{ if } q = a \\ \overline{P \times 2}_{\geq (\hat{0}_P, 1)} \text{ if } q = b \end{cases} .$$

Each fiber has a cone point, so they are all contractible. Hence, f induces a homotopy equivalence by Lemma 3.3.3. ∎

Lemma 3.3.6 (Rambau [72]) *Let P, Q be bounded posets with $\hat{0}_Q \neq \hat{1}_Q$ and distinguished order ideal $J \subseteq P$. Let $f : P \to Q$ be a surjective order-preserving map with order-preserving sections $i, j : Q \to P$. Assume that*

1. $i(Q) \subseteq J$ *and* $j(Q) \subseteq P \setminus J$

2. $(\forall p \in P)\ (i \circ f)(p) \leq p \leq (j \circ f)(p)$

3. $f^{-1}(\hat{0}_Q) \cap J = \{\hat{0}_P\}$ *and* $f^{-1}(\hat{1}_Q) \cap (P \setminus J) = \{\hat{1}_P\}$

Then $\overline{P}$ *is homotopy equivalent to the suspension of* $\overline{Q}$.

Proof: We prove $\overline{P} \simeq \overline{Q \times 2}$. Let $p_1, \ldots, p_t$ be a linear extension of $J - i(Q)$. Let $P_0 = \overline{P}$ and $P_k = P_{k-1} - \{p_k\}$ for $k > 0$. For $k > 0$, $(P_{k-1})_{<p_k}$ is a subset of $i(Q)$ since J is an order ideal of P. Since $p_k \in J$, $(i \circ f)(p_k) \neq \hat{0}_P$ by (3). By (2), $(i \circ f)(p_k) \in (P_{k-1})_{<p_k}$. If $p \in (P_{k-1})_{<p_k}$, then $p \in i(Q)$, so $p = (i \circ f)(p)$ and $(i \circ f)(p) \leq (i \circ f)(p_k)$. Hence, $(P_{k-1})_{<p_k}$ has a maximum element so it is contractible. This implies $P_{k-1} \simeq P_k$, so $\overline{P} \simeq \overline{i(Q) \cup P \setminus J}$.

By a dual argument, we may delete the elements of $(P - J) - j(Q)$ without altering homotopy type. Hence, $\overline{P} \simeq \overline{i(Q) \cup j(Q)} = \overline{Q \times 2}$. ∎

3.3.4 The Crosscut Theorem

A crosscut C of a poset P is a subset of pairwise incomparable elements satisfying the following two conditions.

- For every chain $x_0 < \cdots < x_d$ of P, there exists an element of C comparable to every x_i.

- If $B \subseteq C$, then B has either at most one common minimal upper bound or one common maximal lower bound.

If C is a crosscut of P, then the *crosscut complex* $\Gamma(P, C)$ is the simplicial complex on C containing subsets $B \subseteq C$ for which either $\bigvee B$ or $\bigwedge B$ exists.

For $x \in P$ and subset $C \subseteq P$, let $C_{<x}$ ($C_{>x}$) be the intersection of C with $P_{<x}$ ($P_{>x}$).

Lemma 3.3.7 *If* C *is a crosscut of* P *and* $x \in P$ *such that* $c < x$ *for some* $c \in C$*, then* $C_{<x}$ *is a crosscut of* $P_{<x}$.

Proof: Let T be a chain of $P_{<x}$. Since $T \cup \{x\}$ is a chain of P, there exists an element $c \in C$ such that $T \cup \{x, c\}$ is a chain of P. Then $c < x$ since C is an antichain, so $T \cup \{c\}$ is a chain of $P_{<x}$ and $c \in C_{<x}$. ∎

Proofs of the Crosscut Theorem typically involve the Ideal Relation Theorem or the Nerve Theorem (see [12, Theorem 10.8] or [99, Theorem 7.1]. We prove it by performing a sequence of local moves as in Lemma 3.3.1. Kozlov used a different set of local moves to prove that the crosscut complex on the atoms of a lattice has the same *simple* homotopy type as the proper part of the lattice [56, Theorem 5.2].

Theorem 3.3.8 (Crosscut Theorem) *If C is a crosscut of P, then P is homotopy equivalent to its crosscut complex $\Gamma(P, C)$.*

Proof: Let C be a crosscut of a poset P. The following list is the set of conditions that may be assumed of P and C.

(i) C is the set of atoms of P.

(ii) For $B \subseteq C$, if $\bigwedge B$ exists then $\bigvee B$ exists.

(iii) If B and B' are distinct subsets of C such that $\bigvee B$ and $\bigvee B'$ both exist, then $\bigvee B \neq \bigvee B'$.

(iv) Every element of P is equal to $\bigvee B$ or $\bigwedge B$ for some $B \subseteq C$.

Assume (i),(ii),(iii), and (iv) all hold. Then P is the poset of faces of $\Gamma(P, C)$, so they are homeomorphic.

Assume (ii),(iii), and (iv) hold, but (i) is not true. Assume the Crosscut Theorem holds for posets of size less than $|P|$. Let x be a minimal element of P such that $x < c$ for some $c \in C$. Then $x = \bigwedge C_{>x}$ by (iv) so $\bigvee C_{>x}$ exists by (ii). Hence, the crosscut complex $\Gamma(P_{>x}, C_{>x})$ is a simplex. By Proposition 3.3.1, the posets P and $P - x$ are homotopy equivalent. By (ii) the crosscut complexes $\Gamma(P, C)$ and $\Gamma(P - x, C)$ are identical.

Assume (iii) and (iv) hold but (i) and (ii) do not. Assume the Crosscut Theorem holds for crosscuts of size less than $|C|$. Let B be a minimal subset of C such that $\bigwedge B$

exists but $\bigvee B$ does not. Let P' be an ordering on the set $P \sqcup \{x\}$ agreeing with P such that $x > \bigvee B'$ for any proper subset B' of B. Then $x = \bigvee B$ and $x \neq \bigvee B''$ for any $B'' \subseteq C$ distinct from B. Consequently, C is a crosscut of P' satisfying (iii) and (iv), and $\Gamma(P, C)$ is equal to $\Gamma(P', C)$. If $B = C$, then both P and P' have a minimum element $\bigwedge C$ by (iv), so they are both contractible. Otherwise, $C_{<x}$ is a crosscut of $P'_{<x}$ of size less than $|C|$. Since $\Gamma(P'_{<x}, C_{<x})$ is a simplex, the posets P and P' are homotopy equivalent by Proposition 3.3.1. By induction, we reduce to the case that (ii),(iii), and (iv) hold.

Assume (iv) but not (iii). Let $B \subseteq C$ be a minimal subset such that $\bigvee B = \bigvee B'$ for some other subset $B' \subseteq C$. Let P' be an ordering on the set $P \sqcup \{x\}$ agreeing with P such that $x < \bigvee B'$ and $x > \bigvee B''$ for any proper subset B'' of B. This is well-defined by the minimality of B. The crosscut C of P' still satisfies (iv). The crosscut complexes $\Gamma(P, C)$ and $\Gamma(P', C)$ are identical. Since $\bigvee B'$ is the minimum element of $P'_{>x}$, the posets P and P' is homotopy equivalent by Proposition 3.3.1. By induction, we reduce to the case that (iii) and (iv) hold.

Assume (iv) does not hold. Assume the Crosscut Theorem holds for any poset of size less than $|P|$. Let x be an element of P not equal to $\bigvee B$ or $\bigwedge B$ for any $B \subseteq C$. Up to duality, we may assume there exists $c \in C$ such that $c \leq x$. Since $\bigvee C_{<x} < x$ holds, the crosscut complex $\Gamma(P_{<x}, C_{<x})$ is a simplex, hence contractible. Thus $P_{<x}$ is contractible by the inductive hypothesis. Again, the posets P and $P - x$ are homotopy equivalent and $\Gamma(P, C)$ is isomorphic to $\Gamma(P - x, C)$. Hence, we may reduce to the case where (iv) holds.

Now assume every element of P is of the form $\bigvee B$ or $\bigwedge B$ for some $B \subseteq C$. Assume that there exists a subset $B \subseteq C$ such that $\bigwedge B$ exists but $\bigvee B$ does not. Define $P' := P \sqcup \{x\}$ where $x > \bigvee B'$ for $B' \subseteq B$ whenever $\bigvee B'$ exists. Then $P'_{<x}$ has a crosscut B and its crosscut complex $\Gamma(P'_{<x}, B)$ is a simplex. If $|P'_{<x}| < |P|$, then $P'_{<x}$ is contractible by the induction hypothesis. Otherwise, $P'_{<x} = P$, $B = C$ and $\bigwedge C$ is a cone point of P. By Proposition 3.3.1, the posets P' and P are homotopy equivalent. Hence, we may assume for $B \subseteq C$ if $\bigwedge B$ exists in P, then $\bigvee B$ exists.

Suppose $x \in P$ such that $x < c$ for some $c \in C$. Then $P_{>x}$ has a crosscut $C_{>x}$ and $\bigvee C_{>x}$ exists by assumption. Hence, $\Gamma(P_{>x}, C_{>x})$ is a simplex and Proposition 3.3.1 implies P and $P - x$ are homotopy equivalent.

Now we may assume that C is the set of atoms of P. Suppose there exists distinct subsets B, B' of C such that $\bigvee B = \bigvee B'$. Let $B \subseteq C$ be a subset of minimum size such that there exists $B' \subseteq C$ for which $\bigvee B = \bigvee B'$. Define $P' := P \sqcup \{x\}$ where $x < \bigvee B'$ and $x > \bigvee B''$ whenever B'' is a proper subset of B. Then $P'_{>x}$ has a minimum element $\bigvee B'$, so it is contractible. Hence, P and P' are homotopy equivalent, and they have isomorphic crosscut complexes. Hence, we may assume for $B, B' \subseteq C$ that $\bigvee B \neq \bigvee B'$ whenever both exist.

Putting these assumptions together, P is the face poset of $\Gamma(P, C)$, so they are homotopy equivalent. ■

3.3.5 Other consequences of Lemma 3.3.1

Given a poset P, let P_{nonc} be the subposet of elements x for which $P_{<x}$ is not contractible.

Lemma 3.3.9 $P \simeq P_{\text{nonc}}$.

Proof: Let $p_1, \ldots, p_N$ be a linear extension of P. For each i, let

$$P_i = \{p_j : \ j < i \text{ or } P_{<p_j} \text{ is non-contractible}\}.$$

If $P_{<p_i}$ is non-contractible, then $P_i = P_{i+1}$. Otherwise, $P_i \simeq P_{i+1}$ by Lemma 3.3.1. By induction, we deduce $P \simeq P_{\text{nonc}}$. ■

Recall that $\text{Int}(P)$ is the poset of closed intervals of P, ordered by inclusion. In Lemma 3.3.10 we prove that the interval poset is homeomorphic to the suspension of the original poset. This was originally proved by Walker [100, Theorem 6.1(c)] by specifying a "subdivision map" between geometric realizations of their order complexes. In the spirit of this section, we construct the order complex of $\overline{\text{Int}}(P)$ from $\text{susp}(\overline{P})$ by a sequence of edge-stellations.

Lemma 3.3.10 (Walker [100]) *If P is a bounded poset, $\overline{\text{Int}}(P)$ is homeomorphic to* $\text{susp}(\overline{P})$.

Proof: Let Γ_0 be the complex $\{\hat{0}_P, \hat{1}_P\} * \Delta(\overline{P})$. The edges of this flag simplicial complex are in bijection with proper closed intervals $[x, y]$ of P for which $x \neq y$. Let $I_1, \ldots, I_N$ be a list of the closed intervals with at least two elements such that for $i < j$, $I_i \not\subseteq I_j$. Let $e_1, \ldots, e_N$ be the corresponding list of edges of Γ_0.

For each i, let $\Gamma_i = \mathrm{st}_{e_i}(\Gamma_{i-1})$. We claim that Γ_N is isomorphic to $\Delta(\overline{\mathrm{Int}}(P))$. It is clear that the vertices of Γ_N are in natural bijection with those of $\Delta(\overline{\mathrm{Int}}(P))$. Since order complexes are flag and edge-stellation preserves the flag property, both complexes are flag. It remains to show that they have the same 1-skeleton.

Let I, I' be distinct proper closed intervals of P. If $I \subseteq I'$ then they are adjacent in $\Delta(\overline{\mathrm{Int}}(P))$. We show that they are adjacent in Γ_N.

Let $I = [x, y], I' = [w, z]$ with $w \leq x \leq y \leq z$ and $w < z$. Suppose $I' = I_i$. The set $\{w, x, y, z\}$ is a simplex of Γ_{i-1} since it is a simplex in Γ_0 and none of its edges were subdivided up to Γ_{i-1}. Hence I' is adjacent to both x and y in Γ_i. If $x = y$, then we deduce $I = [x, x]$ and I' are adjacent in Γ_N. If $x < y$ then $I = I_j$ for some $j > i$ and I' is adjacent to both x and y in Γ_{j-1}. Hence, I and I' are adjacent in Γ_j, so they are adjacent in Γ_N.

Now assume I and I' are adjacent in Γ_N. We prove that I and I' are adjacent in $\Delta(\overline{\mathrm{Int}}(P))$.

Since all of the edges of Γ_0 are stellated, we may assume that $|I'| \geq 2$ and that $I' = I_i$ for some i. Let $I = [x, y],\ I' = [w, z]$ for some $x \leq y,\ w < z$. If $x = y$, then x must be adjacent to both w and z in Γ_{i-1}. In particular, x must be comparable to both w and z. Without loss of generality, we may assume $x \leq z$. If $x < w$, then x is not comparable to z in Γ_{i-1}, a contradiction. Hence, $I \subseteq I'$. If $x < y$, then $I = I_j$ for some j. Without loss of generality, we may assume that $i < j$. By the same reasoning as above, we deduce that $x \in [w, z]$ and $y \in [w, z]$. Hence $I \subseteq I'$, as desired. ∎

For a bounded poset P, let $\mathrm{Int}_{\mathrm{nonc}}(P)$ be the poset of closed intervals $[x, y]$ for which (x, y) is non-contractible, ordered by inclusion. We note that if $x = y$ or $x \lessdot y$, then $\Delta((x, y))$ is an empty complex, which is non-contractible.

Lemma 3.3.11 $\overline{\mathrm{Int}}_{\mathrm{nonc}}(P) \simeq \mathrm{susp}(\overline{P})$.

Proof: We prove $\overline{\mathrm{Int}}_{\mathrm{nonc}}(P)$ is homotopy equivalent to $\overline{\mathrm{Int}}(P)$. The result follows from Lemma 3.3.10.

Let $I_1, \ldots, I_N$ be a linear extension of $\overline{\mathrm{Int}}(P)$. For $i \geq 0$, let

$$Q_i = \{I_j : \ j \leq i \text{ or } I_j \text{ is non-contractible}\}.$$

Then $Q_N = \overline{\mathrm{Int}}(P)$ and $Q_0 = \overline{\mathrm{Int}}_{\mathrm{nonc}}(P)$. If I_i is non-contractible then $Q_{i-1} = Q_i$.

Let $i \geq 0$ and assume I_i is contractible. Since $I_j \subseteq I_i$ implies $j \leq i$, the subposet $(Q_i)_{<I_i}$ is equal to $\overline{\mathrm{Int}}(I_i)$. The latter is the suspension of a contractible complex, so it is contractible. Hence, $Q_{i-1} \simeq Q_i$. The result now follows by induction. ∎

Chapter 4

Chambers of real hyperplane arrangements

Let $\mathcal{A}$ be a real central hyperplane arrangement in $\mathbb{R}^n$. Each hyperplane $H \in \mathcal{A}$ divides $\mathbb{R}^n - H$ into two open half spaces, denoted H^+ and H^-. A chamber is (the closure of) a nonempty cone of the form

$$\bigcap_{H \in \mathcal{A}} H^{x(H)}$$

where $x \in \{+,-\}^{\mathcal{A}}$. When $\mathcal{A}$ is a reflection arrangement, a sign vector $x \in \{+,-\}$ defines a chamber if and only if the restriction $x|_{\mathcal{A}'}$ is a chamber of $\mathcal{A}'$ for any rank 2 subarrangement $\mathcal{A}'$. If $\mathcal{A}$ is any arrangement with this rank 2 reduction property, we say it *has the biclosed property.* In this section, we prove that any simplicial or supersolvable arrangement has the biclosed property.

When the chamber poset of an arrangement $\mathcal{A}$ with fundamental chamber c_0 is a lattice, Björner, Edelman, and Ziegler proved that the chambers of $\mathcal{A}$ are in bijection with biconvex sets [14], a weaker property than the bijection with biclosed sets. A non-exhaustive list of chamber posets $\mathrm{Ch}(\mathcal{A}, c_0)$ that are lattices include cases where

1. $\mathcal{A}$ is simplicial ([14] Theorem 3.4),
2. $\mathcal{A}$ is supersolvable and c_0 is incident to a modular flag of intersection subspaces ([14] Theorem 4.6),
3. the rank of $\mathcal{A}$ is at most 3 and c_0 is a simplicial cone ([14] Theorem 3.2), or

4. $\mathcal{A}$ is hyperfactored with respect to c_0 ([54] Theorem 5.1).

Of these examples, the first two have biclosed property, while the third does not. At this time, we are unsure whether biclosed subsets of hyperfactored arrangements correspond to chambers. We suspect that this correspondence does not hold. Our main difficulty is that we do not have a good technique for generating non-supersolvable hyperfactored arrangements.

This section is organized as follows. In §4.2 we prove that simplicial arrangements have the biclosed property. We also compute a formula for joins in a chamber poset of a bineighborly arrangement similar to a formula for joins in the weak order of a Coxeter group given in §2.4.4. We leave the lattice-theoretic consequences of this formula to §5.5.

In §4.3, we prove that supersolvable arrangements have the biclosed property. For supersolvable arrangements, we use this to fill a small gap in the computation of the diameter of the graph of reduced galleries between antipodal chambers, which was given by Reiner and Roichman [81]. We also identify these galleries with another family of biclosed sets. This identification suggests the definition of a *gallery poset*, which we study in §7.

4.1 Bineighborly Arrangements

Let $\mathcal{A}$ be a bineighborly arrangement with respect to a fundamental chamber c_0. Recall that this means for any chamber c, if H, H' are distinct upper walls of c, then $H \cap H'$ is incident to c. Let x be the covector supported by $H \cap H'$ incident to c. If c_1, c_2 are the chambers with $S(c, c_1) = \{H\}$ and $S(c, c_2) = \{H'\}$, then $x \circ (-c_0) = c_1 \vee c_2$ by Corollary 3.2.2. Since x is of codimension 2, the interval $[c, x \circ (-c_0)]$ is a polygon. As $\mathrm{Ch}(\mathcal{A}, c_0)$ is self-dual, this poset is a polygonal lattice.

The join of two arbitrary chambers of $\mathcal{A}$ may be computed by the 2-closure.

Theorem 4.1.1 *If x, y, w are chambers of $\mathcal{A}$ such that $x, y \in [w, -c_0]$ then*

$$S(w, x \vee y) = \overline{S(w, x) \cup S(w, y)}.$$

Consequently, $S(x \vee y) = \overline{S(x) \cup S(y)}$ for chambers $x, y \in \mathrm{Ch}(\mathcal{A}, c_0)$.

Proof: If $w = -c_0$, the claim is immediate. Let $w \in \mathrm{Ch}(\mathcal{A}, c_0)$ and assume that

$$S(w', x \vee y) = \overline{S(w', x) \cup S(w', y)}$$

holds for $x, y \in [w', -c_0]$ whenever $|S(w)| < |S(w')|$.

Let $x, y \in [w, -c_0]$. We may assume $x \wedge y = w$ as otherwise we have

$$\begin{aligned}\overline{S(w,x) \cup S(w,y)} \subseteq S(w, x \vee y) &= S(w, x \wedge y) \cup S(x \wedge y, x \vee y) \\ &= S(w, x \wedge y) \cup \overline{S(x \wedge y, x) \cup S(x \wedge y, y)} \\ &\subseteq \overline{S(w,x) \cup S(w,y)}.\end{aligned}$$

If $x = w$ then $x \leq y$ and the identity

$$S(w, x \vee y) = S(w, y) = \overline{S(w,y)} = \overline{S(w,x) \cup S(w,y)}$$

is clear.

Assume $w < x$ and $w < y$ hold. Let $H \in U(w) \cap S(w,x)$ and $H' \in U(w) \cap S(w,y)$. Let c, c' denote the chambers covering w with $S(w,c) = \{H\}$ and $S(w,c') = \{H'\}$. Since $(\mathcal{A}, c_0)$ is bineighborly, w is incident to $H \cap H'$, so the join $c \vee c'$ satisfies

$$S(w, c \vee c') = \mathcal{A}_{H \cap H'} = \overline{S(w,c) \cup S(w,c')}.$$

The equality

$$S(c \vee c', x \vee y) = \overline{S(c \vee c', x \vee c') \cup S(c \vee c', c \vee y)}$$

holds by the induction hypothesis. The rest of this string of equalities and inequalities follows from properties of closure operators.

$$\begin{aligned}
\overline{S(w,x) \cup S(w,y)} &= \overline{S(w,c) \cup S(c,x) \cup S(w,c') \cup S(c',y)} \\
&= \overline{S(w,c \vee c') \cup S(c,x) \cup S(c',y)} \\
&= \overline{S(c,x) \cup S(c,c \vee c') \cup S(c',y) \cup S(c',c \vee c')} \\
&= \overline{S(c,x \vee c') \cup S(c',c \vee y)} \\
&\supseteq S(w,c \vee c') \cup \overline{S(c \vee c',x \vee c') \cup S(c \vee c',c \vee y)} \\
&= S(w,c \vee c') \cup S(c \vee c',x \vee y) \\
&= S(w,x \vee y)
\end{aligned}$$

∎

Reading defined an arrangement to be *bisimplicial* if $c|_{U(c)}$ is a simplicial cone for all $c \in \mathrm{Ch}(\mathcal{A})$ [75]. We prove in Proposition 4.1.2 that bineighborly arrangements are bisimplicial. This result is somewhat surprising since there exist non-simplicial neighborly polytopes. Not all bisimplicial arrangements define semidistributive lattices nor have joins computed by a 2-closure; see Figure 3.2.

Proposition 4.1.2 *Let $\mathcal{A}$ be an arrangement with fundamental chamber c_0. If $\mathcal{A}$ is bineighborly, then $\mathcal{A}$ is bisimplicial.*

Proof: Let c be a chamber of $\mathcal{A}$, and let $\mathcal{A}' = \mathcal{W}(c) \cap S(c, -c_0)$. By the proof of Theorem 4.1.1, for $I \subseteq \mathcal{A}'$, the set of hyperplanes separating c and

$$\bigvee \{c' : \ (\exists H \in I) \ S(c,c') = \{H\}\}$$

is equal to the 2-closure of I in $\mathcal{A}$. Since every subset of $\mathcal{A}'$ is 2-closed in $\mathcal{A}'$, this implies $\mathcal{A}'$ has $2^{|\mathcal{A}'|}$ distinct chambers, so $c|_{\mathcal{A}'}$ is simplicial. ∎

As remarked in §3.2.2, the graph of reduced galleries between any two chambers is connected. For bineighborly arrangements, we may achieve a stronger result.

Proposition 4.1.3 *Let $\mathcal{A}$ be a bineighborly arrangement with fundamental chamber c_0. If $c, c' \in \mathrm{Ch}(\mathcal{A}, c_0)$ such that $c < c'$, then $\mathrm{Gal}(c,c')$ is $(|U(c) \cap S(c,c')| - 1)$-connected.*

Proposition 4.1.3 is a consequence of the following result of Athanasiadis, Edelman, and Reiner [4, Theorem 2.1]:

Let P be a finite 2-dimensional CW-complex with 1-skeleton G. Endow G with an acyclic orientation so that

1. G has a unique source v_{min} and sink v_{max}, as does its restriction to every 2-face of P,

2. any two faces of P intersect in a unique common face of each, and

3. any two edges of G with a common source lie on a 2-face of P.

Let $\mathrm{Gal}(P)$ be the graph of directed paths from v_{min} to v_{max} supported by G, where two paths are adjacent if they differ only along the boundary of some 2-face of P.

Theorem 4.1.4 (Theorem 2.1 [4]) *If d is the degree of the source of G, then $\mathrm{Gal}(P)$ is $(d-1)$-connected.*

For Proposition 4.1.3, G is the interval $[c, c']$ inside $\mathrm{Ch}(\mathcal{A}, c_0)$. The complex P is formed from G by attaching a 2-dimensional face to intervals $[x \circ c_0, x \circ (-c_0)]$ for $x \in \mathcal{L}_2(\mathcal{A})$ whenever $c \leq x \circ c_0$ and $x \circ (-c_0) \leq c'$. Conditions (1) and (2) hold for any arrangement $\mathcal{A}$. Condition (3) holds by the bineighborly hypothesis. Finally, the degree of the source of G is the number of upper covers of c in $[c, c']$, which is $|U(c) \cap S(c, c')|$.

4.2 Simplicial Arrangements

For $H \in \mathcal{A}$, let $\mathrm{depth}(H)$ be the minimum size of $S(c)$ where $H \in S(c),\ c \in \mathrm{Ch}(\mathcal{A}, c_0)$. Recall that $\mathcal{W}(c)$ denotes the set of all walls of a chamber c.

Proposition 4.2.1 *Let $\mathcal{A}$ be a simplicial arrangement with fundamental chamber c_0, and let $I \subseteq \mathcal{A}$. If I is 2-closed and $I \supseteq \mathcal{W}(c_0)$, then $I = \mathcal{A}$.*

Proof: If $\mathrm{depth}(H) = 1$, then $H \in I$ by assumption.

Let $k > 1$ and suppose $H \in I$ if $\mathrm{depth}(H) < k$. Let $H \in \mathcal{A}$ such that $\mathrm{depth}(H) = k$. Let c be a chamber such that $|S(c)| = k$ and $H \in S(c)$. By minimality of c, H is the

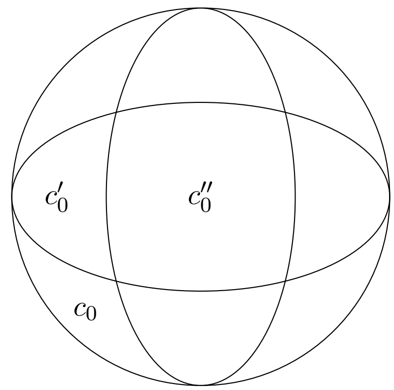


Figure 4.1: $\mathrm{Ch}(\mathcal{A}, c_0)$ is semidistributive and $(\mathcal{A}, c_0)$ is bineighborly. $\mathrm{Ch}(\mathcal{A}, c_0')$ is a non-semidistributive lattice. $(\mathcal{A}, c_0')$ is not bineighborly since c_0'' has two upper walls that do not intersect at the boundary of c_0''. $\mathrm{Ch}(\mathcal{A}, c_0'')$ is not a lattice.

unique wall of c separating it from c_0. Let c' be the chamber with $S(c', c) = \{H\}$, and let H' be any hyperplane in $\mathcal{W}(c') \cap S(d)$. Since c' is simplicial, it is incident to $H \cap H'$.

Let x be the codimension 2 face supported by $H \cap H'$ incident to c'. Let H_1, H_2 be the two walls of $x \circ c_0$ containing x. Let c_1, c_2 be the chambers with $S(x \circ c_0, c_i) = \{H_i\}$ for $i = 1, 2$. Since $|S(c_i)| < k$ for $i = 1, 2$, the depths of H_1 and H_2 are both less than k. Hence, $H_1, H_2 \in I$. Since I is 2-closed, this implies $H \in I$. ∎

Theorem 4.2.2 *Let $\mathcal{A}$ be a simplicial arrangement with fundamental chamber c_0. A subset I of $\mathcal{A}$ is biclosed if and only if $I = S(c)$ for some chamber c.*

Proof: Let $I \subseteq \mathcal{A}$. If $I = S(c)$ for some chamber c, then I is biclosed by Proposition 3.2.9.

Assume I is biclosed. Choose a chamber c minimizing $|I \triangle S(c)|$. Let $I' = I \triangle S(c)$. By Lemma 3.2.10 I' is biclosed with respect to c. The minimality of c implies $\mathcal{W}(c) \subseteq \mathcal{A} - I'$. Since $\mathcal{A} - I'$ is 2-closed with respect to c and it contains $\mathcal{W}(c)$, Proposition 4.2.1 implies I' is empty. ∎

4.3 Supersolvable Arrangements

The following proposition due to Reiner and Roichman was essential to computing the diameter of the graph of reduced galleries in a supersolvable arrangement. Part (2) is implicit in [14, §4].

Proposition 4.3.1 ([81], Proposition 4.6) *Assume that l is a modular line of an arrangement $\mathcal{A}$ with chamber c_0 incident to l. Let $\pi : \mathrm{Ch}(\mathcal{A}, c_0) \to \mathrm{Ch}(\mathcal{A}_l, (c_0)_l)$ be the localization map, and let U denote the fiber $\pi^{-1}(\pi(c_0))$.*

1. *For each hyperplane H of $\mathcal{A} \setminus \mathcal{A}_l$, there exists a unique covector $x \in \mathcal{L}(\mathcal{A})$ with $x^0 = H$ such that some chamber in U is incident to x.*

2. *The fiber $U = \{c_0, c_1, \ldots, c_t\}$ is linearly ordered $c_0 < c_1 < \cdots < c_t$. This induces a linear order $H_1, H_2, \ldots, H_t$ on $\mathcal{A} \setminus \mathcal{A}_l$ such that H_i is the unique hyperplane in $S(c_{i-1}, c_i)$.*

3. *Using the linear order on $\mathcal{A} \setminus \mathcal{A}_l$ from part (2), if $i < j < k$ and if the chamber c_0 incident to l is also incident to $l + H_i \cap H_k$ then $H_j \supseteq H_i \cap H_k$.*

Proof: (1) Fix a hyperplane H in $\mathcal{A} \setminus \mathcal{A}_l$. We begin by proving some chamber of U is incident to H.

Since $-c_0$ is incident to l, there exists a chamber c such that $S(-c_0, c) = \mathcal{A}_l$. Then $S(c_0, c) = \mathcal{A} \setminus \mathcal{A}_l$. By Proposition3.2.1(2), there exists a saturated chain

$$c_0 < c_1 < \cdots < c_t = c$$

in U. Then H is the unique hyperplane separating some adjacent pair of chambers c_{i-1}, c_i.

Let $x, y \in \mathcal{L}(\mathcal{A})$ such that $x^0 = H = y^0$ and x, y are both incident to a chamber in U. Suppose $x(H') = -y(H')$ for some $H' \in \mathcal{A} - \{H\}$. By property (L3) (see 2.3.1), there exists a covector z such that $z(H) = 0 = z(H')$ and $z(H'') = c_0(H'')$ for $H'' \in \mathcal{A}_l$. Since l is modular, there exists a hyperplane in $\mathcal{A}_l$ containing $l + z^0$, contradicting the hypothesis that $z(H'') \neq 0$ for $H'' \in \mathcal{A}_l$.

(2) Let H_i be the unique hyperplane of $\mathcal{A}$ separating c_{i-1} and c_i as defined above. Since $S(c_0, c)$ equals $\mathcal{A} \setminus \mathcal{A}_l$, every hyperplane in $\mathcal{A} \setminus \mathcal{A}_l$ appears exactly once as H_i. For

every i, c_{i-1} and c_i are the only chambers in the fiber U incident to H_i by (1). Hence, $c_0, \ldots, c_t$ are the only chambers in U.

(3) Let $H = l + H_i \cap H_k$. Since c_t is incident to l and $(c_0)_l = (c_t)_l$, H is incident to c_t. Let x and y be the unique covectors incident to c_0 and c_t with $x^0 = y^0 = H$. Since $x(H_i) = -y(H_i)$ there exists $z \in \mathcal{L}(\mathcal{A})$ such that $z(H_i) = 0$ and $z(H') = c_0(H')$ for $H' \in \mathcal{A}_l$. If z^0 is of codimension 3 or more, then $l^0 + z^0$ is of codimension at least 2. But that implies there is some $H' \in \mathcal{A}_l$ besides H for which $z(H') = 0$, an impossibility. Since $z(H) = 0 = z(H_i)$, $z \in \mathcal{L}_2(\mathcal{A})$ and $z^0 = H_i \cap H_k$. The chambers $z \circ c_0$ and $z \circ c_t$ are in the fiber before c_{i-1} and after c_k, respectively. This means they are on opposite sides of H_j, forcing $z(H_j) = 0$. Hence, $H_j \supseteq z^0 = H_i \cap H_k$. ∎

Theorem 4.3.2 *Let $\mathcal{A}$ be a supersolvable arrangement with fundamental chamber c_0 incident to a modular flag.*

1. *For $I \subseteq \mathcal{A}$, there exists a chamber $c \in \mathrm{Ch}(\mathcal{A})$ with $I = S(c_0, c)$ if and only if I is biclosed.*

2. *For $c, d \in \mathrm{Ch}(\mathcal{A})$, the separation set $S(c \vee d)$ is the 2-closure of $S(c) \cup S(d)$.*

Proof: Let $l \in L(\mathcal{A})$ be a modular line incident to c_0 such that $(c_0)_l$ is incident to a modular flag of $\mathcal{A}_l$. Assume both parts of the theorem hold for the pair $(\mathcal{A}_l, (c_0)_l)$.

(1) Let $I \subseteq \mathcal{A}$. If $I = S(c_0, c)$ for some chamber c, then I is biclosed by Proposition 3.2.9.

Assume I is biclosed. The restriction $I \cap \mathcal{A}_l$ is $(c_0)_l$-biclosed, so there exists a chamber $\overline{c} \in \mathrm{Ch}(\mathcal{A}_l)$ such that $S((c_0)_l, \overline{c}) = I \cap \mathcal{A}_l$. Since c_0 is incident to l, there exists a chamber $c \in \mathrm{Ch}(\mathcal{A})$ such that $S(c_0, c) = I \cap \mathcal{A}_l$ by Proposition 3.2.1(1). Let $c_1, \ldots, c_{t+1} \in \mathrm{Ch}(\mathcal{A})$ such that $c = c_1$, $S(c_1, c_{t+1}) = \mathcal{A} - \mathcal{A}_l$, $(c_i)_l = \overline{c}$ and $|S(c_i, c_{i+1})| = 1$ for all i. Let $H_1, \ldots, H_t$ be the hyperplanes of $\mathcal{A} - \mathcal{A}_l$ where $S(c_i, c_{i+1}) = \{H_i\}$ for all i.

Assume $H_i \in I$ for some $i > 1$, and let $X = H_{i-1} \cap H_i$. Since l is modular, $X + l$ is a hyperplane of $\mathcal{A}$ containing l. If $X + l \in S(c_0, c)$ then H_{i-1} is in the 2-closure of $\{X + l, H_i\} \subseteq I$. If $X + l \notin S(c_0, c)$ then H_i is in the 2-closure of $\{X + l, H_{i-1}\}$. Since I is biclosed, both cases imply $H_{i-1} \in I$. Hence, $I \cap (\mathcal{A} - \mathcal{A}_l)$ is an initial segment of hyperplanes $H_1, \ldots, H_k$ for some k, so $I \cap (\mathcal{A} - \mathcal{A}_l) = S(c, c_{k+1})$ holds. Therefore, we obtain

$$I = S(c_0, c) \cup S(c, c_{k+1}) = S(c_0, c_{k+1}).$$

(2) Let $c, d \in \mathrm{Ch}(\mathcal{A})$. The equality

$$S((c \vee d)_l) = \overline{S(c_l) \cup S(d_l)}$$

holds by the assumption on $(\mathcal{A}_l, (c_0)_l)$. Since c_0 is incident to l, there exist chambers $b, c', d' \in \mathrm{Ch}(\mathcal{A})$ incident to l such that $b_l = (c \vee d)_l$, $c'_l = c_l$, $d'_l = d_l$ by Proposition 3.2.1(1). The above equality then lifts to

$$S(b) = \overline{S(c') \cup S(d')}.$$

Let $c_1, \ldots, c_{t+1} \in \mathrm{Ch}(\mathcal{A})$ such that $c = c_1$, $S(c_1, c_{t+1}) = \mathcal{A} - \mathcal{A}_l$, $(c_i)_l = \overline{b}$ and $|S(c_i, c_{i+1})| = 1$ for all i. Let $H_1, \ldots, H_t$ be the hyperplanes of $\mathcal{A} - \mathcal{A}_l$ where $S(c_i, c_{i+1}) = \{H_i\}$ for all i.

The join $c \vee d$ is equal to c_{k+1} for some k. Since H_k is a wall of $c \vee d$, either $H_k \in S(c)$ or $H_k \in S(d)$. By symmetry, we may assume $H_k \in S(c)$. Suppose there exists H_i with $i < k$, and let $X = H_i \cap H_k$, $H = X + l$. If $H \in S(b)$ then H_i is in the 2-closure of $\{H, H_k\}$, so it lies in $\overline{S(c) \cup S(d)}$. If $H \notin S(b)$, then $H \notin S(c)$ and H_k is in the 2-closure of $\{H_i, H\}$. Since $S(c)$ is biclosed, this forces $H_i \in S(c)$. ∎

The following corollary follows immediately from Theorem 4.3.2 and Proposition 3.2.6.

Corollary 4.3.3 *Let $\mathcal{A}$ be a supersolvable arrangement with fundamental chamber c_0. For any admissible permutation $H_1, \ldots, H_N$ of $\mathcal{A}$ there exists a reduced gallery $c_0, \ldots, c_N$ such that $S(c_{i-1}, c_i) = \{H_i\}$.*

We now fill in the gap in the proof of [81, Theorem 1.1] by adapting that proof in our language, and applying Corollary 4.3.3 with Lemma 3.2.7.

Theorem 4.3.4 ([81], Theorem 1.1) *Let $\mathcal{A}$ be a rank n supersolvable arrangement with fundamental chamber c_0 incident to a modular flag. The graph of reduced galleries from c_0 to $-c_0$ has diameter $|L_2(\mathcal{A})|$.*

Proof: Let $X_1 \supsetneq \cdots \supsetneq X_{n-1}$ be a modular flag such that c_0 is incident to $X_i \in L_i(\mathcal{A})$ for all i. Let $x_1, \dots, x_{n-1}$ be the unique covectors of $\mathcal{A}$ with $c_0 \geq x_1 \geq \cdots \geq x_{n-1}$ and $x_i^{-1}(0) = \mathcal{A}_{X_i}$ for all i. We set $l = X_{n-1}$. Let r_0 be a maximal chain in $\mathrm{Ch}(\mathcal{A}, c_0)$ extending the chain $x_1 \circ (-c_0) < \cdots < x_{n-1} \circ (-c_0)$. By Proposition 3.2.14, it suffices to show that r_0 is L_2-accessible. We proceed by induction on n.

Let r be a reduced gallery from c_0 to $-c_0$ and assume $L_2(r_0, r) \subseteq L_2(\mathcal{A}_l)$. If $H \in \mathcal{A}_l$, $H' \in \mathcal{A} - \mathcal{A}_l$ then $(r_0)_{H \cap H'} = r_{H \cap H'}$ and $(r_0)_{H \cap H'}$ crosses H before H'. Hence, r must contain the chamber $x_{n-1} \circ (-c_0)$. By Proposition 4.3.1(2), the interval $[x_{n-1} \circ (-c_0), -c_0]$ of $\mathrm{Ch}(\mathcal{A}, c_0)$ is a chain, so the galleries r and r_0 agree above x_{n-1}. Since $[c_0, x_{n-1} \circ (-c_0)]$ is isomorphic to $\mathrm{Ch}(\mathcal{A}_l, (c_0)_l)$, the distance between r_0 and r in the reduced gallery graph is equal to $|L_2(r_0, r)|$ by the induction hypothesis.

Now assume $L_2(r_0, r) \not\subseteq L_2(\mathcal{A}_l)$. Let $H_1, \dots, H_N$ be the total order on $\mathcal{A}$ induced by r, and let k be the smallest index for which $H_k \supseteq l$ but $H_{k-1} \not\supseteq l$. Let $X = H_{k-1} \cap H_k$. Let c_1 be the largest chamber in r incident to x_{n-1}. By the assumption on k, $(c_1)_l$ is incident to H_k. Since c_1 is incident to l, this implies c_1 is incident to H_k. If i, j are indices with $i < j < k$ such that $H_i \cap H_k = X$ holds, then $H_j \supseteq X$ by Proposition 4.3.1(3). By Corollary 4.3.3, the gallery r is incident to X, so there exists a gallery r' adjacent to r such that $|L_2(r_0, r')| = |L_2(r_0, r)| - 1$. By induction, there exists a gallery r'' of distance $|L_2(r_0, r)| - |L_2((r_0)_l, r_l)|$ from r such that $L_2(r_0, r'') \subseteq L_2(\mathcal{A}_l)$. By the previous case, this implies the distance between r_0 and r is equal to $|L_2(r_0, r)|$. ∎

Chapter 5

Crosscut-simplicial lattices

Many familiar posets have a Möbius function that only takes values in the set $\{1, -1, 0\}$. To explain this occurrence, Hersh and Mészáros introduced SB-labelings, a labeling of the covering relations of a lattice which ensures that every interval is either contractible or homotopy equivalent to a sphere [47]. Hersh and Mészáros proved that lattices with SB-labelings are crosscut-simplicial. We say a lattice is *crosscut-simplicial* if for any interval $[x, y]$, the join of any proper subset of atoms of $[x, y]$ is not equal to y; see Figure 5.1. Equivalently, a lattice is crosscut-simplicial if the crosscut complex on the atoms of any nuclear interval is the boundary of a simplex. In particular, a crosscut-simplicial lattice has every interval either contractible or homotopy equivalent to a sphere.

We define SB-labelings in §5.1 and give some examples. One advantage of the crosscut-simplicial property over SB-labeling is its behavior under standard lattice constructions, which we cover in §5.2.

A large family of crosscut-simplicial latties are the meet-semidistributive lattices

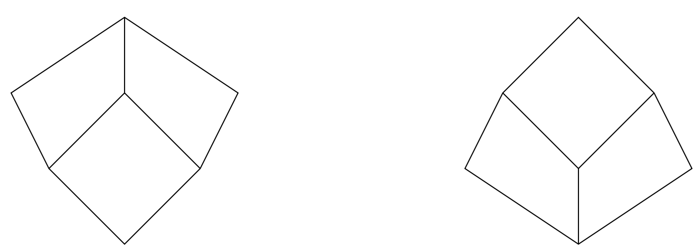


Figure 5.1: The lattice on the left is crosscut-simplicial, while the lattice on the right is not.

(§5.3). Edelman and Walker proved that every interval of any chamber poset $\mathrm{Ch}(\mathcal{A}, c_0)$ is either contractible or homotopy equivalent to a sphere [36]. We prove that $\mathrm{Ch}(\mathcal{A}, c_0)$ is a crosscut-simplicial lattice exactly when it is meet-semidistributive in §5.5.

5.1 SB-labelings

An *SB-labeling* of a lattice is a labeling λ of the covering relations such that

(SB1) if y and z are distinct elements covering some element x, then $\lambda(x \lessdot y)$ is distinct from $\lambda(x \lessdot z)$; and

(SB2) if B is a subset of atoms of $(x, \hat{1})$, then every saturated chain from x to $\bigvee B$ contains only labels in the set $\{\lambda(x \lessdot y) : \ y \in B\}$, and each of those labels occurs at least once.

Example 5.1.1 ([47] Theorem 5.1) *Let L be a finite distributive lattice. Then L is isomorphic to $\mathcal{O}(J(L))$, the poset of order ideals of its subposet of join-irreducibles. If X, Y are order ideals of $J(L)$ such that $X \lessdot Y$ in $\mathcal{O}(J(L))$, then there is a unique element in $Y - X$. Define $\lambda : \mathrm{Cov}(L) \to J(L)$ such that $Y - X = \{\lambda(X \lessdot Y)\}$ for $(X, Y) \in \mathrm{Cov}(L)$. Property (SB1) clearly holds for this λ.*

If $A \subseteq J(L)$ such that $X \cup \{a\}$ is an order ideal for all $a \in A$, then $X \cup A$ is an order ideal, so $X \cup A = \bigvee_{a \in A} X \cup \{a\}$. Every maximal chain in $[X, X \cup A]$ contains each of the labels in A exactly once. Hence, (SB2) is satisfied.

In a similar way, any join-distributive lattice inherits an SB-labeling from an associated convex geometry [67, Theorem 1.1].

The topological significance of an SB-labeling is encapsulated in the following theorem.

Theorem 5.1.2 ([47] Theorem 3.7) *If L admits a SB-labeling, then L is crosscut-simplicial.*

The converse need not hold, as shown in Figure 5.2. However, this example does have an SB-labeling if one relaxes condition (2) by

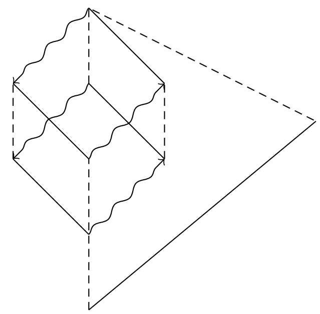


Figure 5.2: A lattice with an edge-labeling satisfying (SB1) and (SB2′) but not (SB2).

(SB2′) if B is a subset of atoms A of $(x, \hat{1})$, then every saturated chain from x to $\bigvee B$ contains each label in the set $\{\lambda(x \lessdot y) : \ y \in B\}$ at least once, and it contains no labels from the set $\{\lambda(x \lessdot z) : \ z \in A - B\}$.

We call an edge-labeling satisfying (SB1) and (SB2′) a *weak SB-labeling.*

Theorem 5.1.3 *If L admits a weak SB-labeling, then it is crosscut-simplicial.*

Proof: Let λ be a weak SB-labeling of L. Let $[x, y]$ be a nuclear interval of L, and let B be a subset of atoms of $[x, y]$. If z is an atom of $[x, y]$ such that $z \notin B$, then the label $\lambda(x \lessdot z)$ is not in any maximal chain of $[x, \bigvee B]$. In particular, $z \not\leq \bigvee B$, so $\bigvee B < y$. ∎

We do not know of a crosscut-simplicial lattice that does not admit a weak SB-labeling.

Example 5.1.4 *The weak order of a finite Coxeter system was originally proved to be crosscut-simplicial by Björner [10], [13, Theorem 3.2.7]. The weak order inherits an SB-labeling from its Cayley graph [47, Theorem 5.3]. Namely, for covering relations $u \lessdot v$ in the weak order on (W, S), set $\lambda(u, v) = u^{-1}v$, which is an element of S.*

As usual, (SB1) is easy to verify: if $(u, v), (u, v') \in \mathrm{Cov}(W)$ with $\lambda(u, v) = \lambda(u, v')$, then $u^{-1}v = u^{-1}v'$ and $v = v'$. For (SB2), if $u \in W$ and $J \subseteq \mathrm{Asc}(u)$, then $\bigvee_{s \in J} us = uw_0(J)$. The interval $[u, uw_0(J)]$ is isomorphic to $[e, w_0(J)]$ by the label-preserving isomorphism $v \mapsto u^{-1}v$. The sequence of labels of a maximal chain in $[e, w_0(J)]$ is a reduced word for $w_0(J)$. But every reduced word for $w_0(J)$ contains every letter in J and none of the letters in $S - J$. Hence, (SB2) is satisfied.

The weak order has another natural labeling. Let T be the set of all reflections of the Coxeter system (W,S). For $u \lessdot v$, set $\lambda_T(u,v) = uv^{-1}$, which is an element of T. Once again, (SB1) follows from a simple computation. It is easy to check that (SB2) does not hold for λ_T (e.g., when (W,S) is the type A_2 Coxeter system).

We prove (SB2′) holds. Let $u \in W$ and $J \subseteq \mathrm{Asc}(u)$, so $\bigvee_{s\in J} us = uw_0(J)$. Then $\mathrm{Inv}(uw_0(J)) = \mathrm{Inv}(u) \cup \overline{\{usu^{-1} :\ s \in J\}}$. As every maximal chain in $[u, uw_0(J)]$ has every element of $\mathrm{Inv}(uw_0(J)) - \mathrm{Inv}(u)$ as a label exactly once, any maximal chain contains the set $\{usu^{-1} :\ s \in J\}$ as labels. If $t \in T - J$ such that $u \lessdot tu$ then $\mathrm{Inv}(u) \cup \{t\}$ is a biclosed set, so $t \notin \overline{\{usu^{-1} :\ s \in J\}}$.

The Tamari lattice also admits an SB-labeling [47, Theorem 5.5]. Currently, it is not known whether other Cambrian lattices admit an SB-labeling, though we may deduce from Theorem 5.4.1 that Cambrian lattices do have a weak SB-labeling.

5.2 Constructions

The following theorem is essentially a restatement of Corollary 2.4 of [75].

Theorem 5.2.1 *Let L be a lattice with lattice congruence Θ. The crosscut complex of any interval of L/Θ is isomorphic to the crosscut complex of some interval of L.*

Proof: Let $([x],[y])$ be an interval of L/Θ. Let A be the set of atoms of $(\pi^{\uparrow}(x), \pi^{\uparrow}(y))$. Let y' be the smallest element Θ-equivalent to y such that $\bigvee A \leq y'$. We claim that the crosscut complex of $(\pi^{\uparrow}(x), y')$ is isomorphic to that of $([x],[y])$.

If $\bigvee A < y'$, then $[\bigvee A] < [y]$, and both complexes are isomorphic to a $(|A|-1)$-simplex. Thus, we may assume $\bigvee A = y'$.

By Lemma 3.1.6, the map $a \mapsto [a]$ is a bijection on the sets of atoms of $(\pi^{\uparrow}(x), y')$ and $([x],[y])$. Let $B \subseteq A$. If $\bigvee B = y'$, then $\bigvee_{b\in B}[b] = [\bigvee B] = [y]$.

Conversely, suppose $\bigvee_{b\in B}[b] = [y]$ and assume $\bigvee B < y'$. Then there exists $a \in A - B$ such that $\bigvee B < a \vee (\bigvee B)$ since $\bigvee B < \bigvee A$. Since a covers $\pi^{\uparrow}(x)$, this forces $a \wedge (\bigvee B) = \pi^{\uparrow}(x)$. But, $[x] < [a] < [\bigvee B]$ so $[a] \wedge [\bigvee B] \neq [x]$, a contradiction. ∎

Corollary 5.2.2 *Let L be a crosscut-simplicial lattice. If Θ is a lattice congruence of L, then the quotient L/Θ is crosscut-simplicial.*

The projection $\pi : P[C] \to P$ defined by $\pi(x, \epsilon) = x$ is order-preserving. If P is a lattice, the map π is a lattice quotient map. The crosscut complexes of a doubled lattice are related to those of the original lattice as in the following proposition. We let $\Delta(A)$ denote the simplicial complex of all subsets of A. If Γ is a simplicial complex and B a subset of the ground set, we let $\Gamma|_B$ denote the induced subcomplex on B.

Proposition 5.2.3 *Let C be an order-convex subset of a lattice L. Let I be an open interval of $L[C]$ and let A be the set of atoms of $\pi(I)$. At least one of the following holds.*

1. $\Gamma(I) \cong \Gamma(\pi(I))$
2. $\Gamma(I) \cong \Delta(A)$
3. $\Gamma(I) \cong \{v\} * \Gamma(\pi(I))|_{A \cap C}$
4. $\Gamma(I) \cong \Delta(A) \cup (\{v\} * \Gamma(\pi(I)))$

Proof: (of Proposition 5.2.3) We divide the possible intervals of $L[C]$ into four cases. Let $I = ((x, \epsilon), (y, \epsilon'))$ be an interval. Then exactly one of the following holds:

1. $\epsilon = \epsilon'$ or $x, y \notin C$,
2. $\epsilon < \epsilon',\ x \notin C$, and $y \in C$,
3. $\epsilon < \epsilon',\ x \in C$, and $y \notin C$, or
4. $\epsilon < \epsilon', x \in C$, and $y \in C$.

We verify that these line up with the four cases for $\Gamma(I)$ listed above.

(1) If $\epsilon = \epsilon'$, then the open interval $((x, \epsilon), (y, \epsilon'))$ is isomorphic to (x, y). If $\epsilon < \epsilon'$ and x, y are both not in C, then the join of some atoms B in $((x, 0), (y, 1))$ is equal to $(y, 1)$ if and only if the join of $\{\pi(b) : b \in B\}$ equals y. In both cases, the crosscut complexes of I and $\pi(I)$ are isomorphic.

(2) If $\epsilon < \epsilon',\ x \notin C$, and $y \in C$, then the join of all of the atoms of I is bounded above by $(y, 0)$. Hence, $\Gamma(I)$ is a simplex.

(3) Suppose $\epsilon < \epsilon'$, $x \in C$, and $y \notin C$. Then

$$\{(x,1)\} \cup \{(a,0) : \ a \in A \cap C\}$$

is the set of atoms of I. If B is a set of atoms of I whose join is equal to (z, ϵ''), then $\bigvee(B \cup \{(x,1)\})$ equals $(z,1)$. If $(z,\epsilon'') < (y,1)$ then $(z,1) < (y,1)$ since $y \notin C$. If $(z,\epsilon'') = (y,1)$, then $\bigvee_{b \in B} \pi(b) = y$. Hence, $\Gamma(I)$ is the cone $\{(x,1)\} * \Gamma(\pi(I))|_{A \cap C}$.

(4) If $\epsilon < \epsilon', x \in C$, and $y \in C$, then I is isomorphic to $\overline{[x,y] \times 2}$. Let $\phi : A \to I$ be the inclusion $a \mapsto (a,0)$. The atom set of I is $\{(x,1)\} \cup \phi(A)$. Since $(a,0) \leq (y,0)$ for $a \in A$, the deletion $\Gamma(I) - \{(x,1)\}$ is equal to $\Delta(\phi(A))$. If $A' \subseteq A$ then $(x,1) \vee \bigvee \phi(A)$ equals $(\bigvee A, 1)$, so the link of $(x,1)$ is equal to $\phi(\Gamma(\pi(I)))$. ∎

Corollary 5.2.4 *Let L be a lattice with an order convex subset C. If every interval of L is either contractible or homotopy equivalent to a sphere, then the same holds for $L[C]$. In particular, if L is congruence-normal, then every interval of L is either contractible or homotopy equivalent to a sphere.*

Proof: Let I be a closed interval of L. It suffices to show that the four constructions for $\Gamma(\overline{I})$ in Proposition 5.2.3 preserve the property of being contractible or homotopy equivalent to a sphere. The first two cases are trivial. The third case is a cone, so it is contractible. In the fourth case, I is isomorphic to $\pi(I) \times 2$, so $\overline{I}$ is homeomorphic to the suspension of $\overline{\pi(I)}$ by Theorem 5.1(d) of [100]. ∎

A finite lattice is distributive if and only if it may be obtained from the one-element lattice by a sequence of doublings at principal order filters. While not every congruence-normal lattice is crosscut-simplicial, we deduce from Proposition 5.2.3 that some doublings preserve the crosscut-simplicial property, as described in the following corollary. An example is given in Figure 5.3.

Corollary 5.2.5 *Let C be an order-convex subset of a crosscut-simplicial lattice L. If for $x \in C$, $y \in L - C$, $x \leq y$ the interval $[x,y]$ contains an atom not in C, then $L[C]$ is crosscut-simplicial. In particular, if C is an order filter, then $L[C]$ is crosscut-simplicial.*

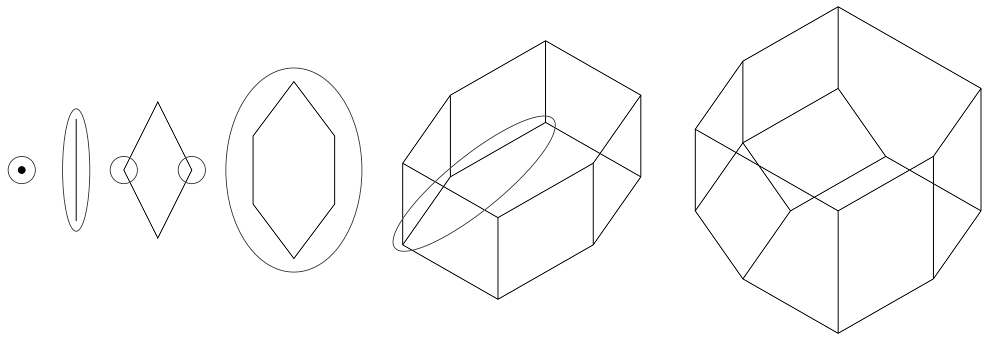


Figure 5.3: A sequence of doublings at order-convex sets satisfying the conditions of Corollary 5.2.5. The final poset is the quotient lattice of Figure 3.1.

Proof: Let $I = [(x, \epsilon), (y, \epsilon')]$ be an interval of $L[C]$. If I is an interval of type (1),(2), or (4) in Proposition 5.2.3, then $\Gamma(I)$ is either a simplex or its boundary. If I is of type (3), then $x \in C$, $y \in L - C$, and $\epsilon < \epsilon'$. Let A be the set of atoms of $[x, y]$. By assumption, $A \cap C$ is a proper subset of C. Since $\Gamma([x, y])$ is either a simplex or its boundary, the restricted complex $\Gamma([x, y])|_{A\cap C}$ is a simplex. Therefore, $\Gamma(I)$ is isomorphic to the simplex $\{v\} * \Gamma([x, y])|_{A\cap C}$. ∎

5.3 Semidistributive lattices

Theorem 5.3.1 *If L is a finite meet-semidistributive lattice, then L is crosscut-simplicial.*

Proof: Let A be the set of atoms of L. Since meet-semidistributivity is inherited by intervals, it suffices to prove $\bigvee B < \bigvee A$ whenever B is a proper subset of A.

We proceed by induction on $|A|$. Let B be a minimal subset of A such that $\bigvee B = \bigvee A$, and let $x \in B$. If $A - x$ contains an element z such that z and $\bigvee(B - x)$ are incomparable, then $x \wedge z = \hat{0} = \bigvee(B - x) \wedge z$ holds. But this implies $z = (x \vee \bigvee(B - x)) \wedge z = \hat{0}$, a contradiction. Hence, $A - x$ is the set of atoms of the meet-distributive lattice $[\hat{0}, \bigvee(B - x)]$. By induction, we have $A - x = B - x$, as desired. ∎

Corollary 5.3.2 *Every interval of a meet-semidistributive lattice or join-semidistributive lattice is either contractible or homotopy equivalent to a sphere.*

Figure 5.2 shows an example of a crosscut-simplicial lattice that is not meet-semidistributive. A join-semidistributive lattice may not be crosscut-simplicial, but its crosscut complex still admits a simple discription.

Proposition 5.3.3 *If L is a join-semidistributive lattice with atom set A, then its crosscut complex is either a $(|A|-1)$-simplex or a pure $(|A|-2)$-subcomplex of the $(|A|-1)$-simplex.*

Proof: If $\bigvee A < \hat{1}$, then the crosscut complex of L is a $(|A|-1)$-simplex. Hence, we may assume $\bigvee A = \hat{1}$. We prove that the maximal faces of the crosscut complex are all of dimension $|A|-2$.

Let B be a maximal subset of A such that $\bigvee B < \hat{1}$. Suppose $A - B$ has two distinct elements x, y. By the maximality of B, one has $x \vee (\bigvee B) = \hat{1} = y \vee (\bigvee B)$. But this implies $(x \wedge y) \vee (\bigvee B) = \hat{1}$, which is impossible since $(x \wedge y) \vee (\bigvee B) = \bigvee B < \hat{1}$. ∎

5.4 CU-labelings are SB-labelings

The second labeling in Example 5.1.4 is an example of a CU-labeling. We prove that CU-labelings are always weak SB-labelings.

Theorem 5.4.1 *If L is a congruence-uniform lattice with a CU-labeling λ, then λ is a weak SB-labeling.*

Proof: Let $x \in L$, and let $y, z \in L$ such that $(x,y), (x,z) \in \mathrm{Cov}(L)$ and $\lambda(x,y) = \lambda(x,z)$. Let j be the unique join-irreducible such that $\lambda(j_*, j) = \lambda(x,y)$. Then $j \vee x = y$ and $j \vee x = z$, so $y = z$. Hence, (SB1) is satisfied by λ.

Now let $x \in L$, $A = \{y \in L : (x,y) \in \mathrm{Cov}(L)\}$ and $B \subseteq A$. For $y \in A$, let j^y be the join-irreducible with $\lambda(j^y_*, j^y) = \lambda(x,y)$. Then $\bigvee_{y \in B} y = x \vee \bigvee_{y \in B} j^y$.

Let $y \in B$ and let $x_0 < \cdots < x_N$ be a maximal chain in $[x, \bigvee_{y \in B} y]$. Let $i > 0$ be the smallest index for which $y \leq x_i$. Then $y \vee x_{i-1} = x_i$ and $y \wedge x_{i-1} = x$. Hence

$\lambda(x,y) = \lambda(x_{i-1}, x_i)$. Conversely, suppose $z \in A$ and $\lambda(x,z) = \lambda(x_{j-1}, x_j)$ for some j. Then $z = j^z \vee x \leq j^z \vee x_{j-1} = x_j \leq \bigvee_{y \in B} y$ holds. Since congruence-uniform lattices are semidistributive, L is crosscut-simplicial. Hence, $z \in B$. This completes the proof of (SB2′). ∎

5.5 Bineighborly arrangements

Our main result in this section is that the poset of chambers is a crosscut-simplicial lattice if and only if it is bineighborly. Nathan Reading proved furthermore that the bineighborly property is equivalent to the poset of chambers being a polygonal lattice [79, Theorem 1-6.10].

Theorem 5.5.1 *Let $\mathcal{A}$ be a real, central hyperplane arrangement with fundamental chamber c_0. The following are equivalent.*

1. $\mathrm{Ch}(\mathcal{A}, c_0)$ *is crosscut-simplicial.*

2. $(\mathcal{A}, c_0)$ *is bineighborly.*

3. $\mathrm{Ch}(\mathcal{A}, c_0)$ *is a semidistributive lattice.*

Proof: (of Theorem 5.5.1) We show (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (1) is a special case of Theorem 5.3.1.

(1) $\Rightarrow$ (2): Suppose $\mathrm{Ch}(\mathcal{A}, c_0)$ is crosscut-simplicial. Let H and H' be upper walls of a chamber c. Let $d, d' \in \mathrm{Ch}(\mathcal{A})$ with $S(c,d) = \{H\}$, $S(c,d') = \{H'\}$. Since $\mathrm{Ch}(\mathcal{A}, c_0)$ is crosscut-simplicial, d and d' are the only atoms of the interval $[c, d \vee d']$. Thus, the set $S(c, d \vee d')$ contains no walls of c besides H and H'. Let α be a generic point in the intersection of the cones $c|_{\mathcal{W}(c)}$ and $d|_{\mathcal{W}(c)}$, and let β be a generic point in the intersection of $d'|_{\mathcal{W}(c)}$ and $(d \vee d')|_{\mathcal{W}(c)}$. Both α and β are points in H separated by H' and no other wall of c. The line segment between α and β intersects H', so there is a point in $H \cap H'$ on the same side as c of any $H'' \in \mathcal{W}(c) - \{H, H'\}$. Hence, $c|_{\mathcal{W}(c)}$ is incident to $H \cap H'$. Since the face posets of c and $c|_{\mathcal{W}(c)}$ are the same, the chamber c is incident to $H \cap H'$.

(2) $\Rightarrow$ (3): Assume $\mathcal{A}$ is bineighborly. Let $c \in \mathrm{Ch}(\mathcal{A}),\ H, H' \in U(c)$, and let a, b be the chambers with $S(c,a) = \{H\},\ S(c,b) = \{H'\}$. By the bineighborly assumption, c is incident to $H \cap H'$. Hence, by Proposition 3.2.1(4), there exists a chamber c' such that $S(c,c') = \mathcal{A}_{H\cap H'}$. If d is some chamber such that $H, H' \in S(c,d)$, then $\mathcal{A}_{H\cap H'} \subseteq S(c,d)$. Therefore, c' is the join of a and b. By Lemma 3.1.1, this implies $\mathrm{Ch}(\mathcal{A}, c_0)$ is a lattice.

It remains to prove the following claim.

Claim: For $a \leq b$ and $x, y \in [a,b],\ z \in \mathrm{Ch}(\mathcal{A}, c_0)$, if $x \wedge z = y \wedge z$, then $(x \vee y) \wedge z = x \wedge z$.

If $a = b$ or $a \lessdot b$, the claim is trivial. Let $a < b$ and suppose the claim holds for all proper subintervals of $[a,b]$. Let x, y, z be chambers such that $x \wedge z = y \wedge z,\ x, y \in [a,b]$. We may assume $a = x \wedge y$ and $x \vee y = b$ by the inductive hypothesis. We have

$$x \wedge z = x \wedge (x \wedge z) = x \wedge (y \wedge z) = a \wedge z.$$

Let u be a coatom of $[x,b]$. Since $x \leq u < b = x \vee y$ holds, u is not an upper bound for y. We have $a = x \wedge y \leq u \wedge y$. Since $x \vee (u \wedge y) \leq u < b$ and $u \wedge y \in [a,y]$, the inductive hypothesis implies $(x \vee (u \wedge y)) \wedge z = x \wedge z$. If $a < u \wedge y$ then the inductive hypothesis implies $((x \vee (u \wedge y)) \vee y) \wedge z = x \wedge z$. This simplifies to $(x \vee y) \wedge z = x \wedge z$, as desired. Thus, we may assume $a = u \wedge y$ holds for every coatom u of $[x,b]$.

Let d be an atom of $[a,y]$. If u is a coatom of $[x,b]$, then d and u are incomparable, so $S(a,d) = S(u,b) = \{H\}$ for some hyperplane H. In particular, H is the unique hyperplane in $S(a,y) \cap U(a)$ and in $S(x,b) \cap U(b)$.

Let c be an atom of $[a,x]$ and let H' be the hyperplane separating c and a. By assumption, a is incident to $H \cap H'$, so by Proposition 3.2.1(4) the join of c and d is the chamber satisfying $S(a, c \vee d) = \mathcal{A}_{H\cap H'}$.

Since $c \in [a,x],\ d \in [a,y]$, we have

$$c \wedge z = a \wedge z = d \wedge z.$$

Assume $c \vee d = b$. Then $S(a \wedge z, b \wedge z) \subseteq S(a,b) = \mathcal{A}_{H\cap H'}$ holds. Neither H nor H' is in $S(a \wedge z, b \wedge z)$ since $c \wedge z = a \wedge z = d \wedge z$. Hence, $a \wedge z = b \wedge z$.

Now assume $c \vee d < b$. Since x and $c \vee d$ are both in $[c,b]$, the equality $x \wedge z = (x \vee (c \vee d)) \wedge z$ holds by induction. Similarly, $y \wedge z = (y \vee c \vee d) \wedge z$. Finally, $x \vee (c \vee d)$

and $y \vee (c \vee d)$ are both elements of $[c \vee d, b]$, so

$$x \wedge z = ((x \vee (c \vee d)) \vee (y \vee (c \vee d))) \wedge z = (x \vee y) \wedge z.$$

$\blacksquare$

Chapter 6

Higher Bruhat orders

The Higher Bruhat order $\mathrm{HB}(n,d)$ is a poset of biclosed subsets of $\binom{[n]}{d+1}$ introduced by Manin and Schechtman (Figure 1.7). To match the previous literature, we use the term *consistent set* rather than biclosed set in this context. The higher Bruhat orders have many equivalent interpretations, including single-element extensions of an alternating matroid, cubical tilings of a cyclic zonotope, and "admissible" permutations of $\binom{[n]}{d}$ up to a suitable equivalence; see [102, Theorem 4.1], [64], or [55]. We recall these definitions in §6.1.

Consistent subsets of $\binom{[n]}{3}$ are in natural bijection with simple pseudoline arrangements with n pseudolines, cyclically ordered at infinity. We also identify one of the two infinite regions bounded by 1 and n as the "bottom" region. The consistent set associated to a simple pseudoline arrangement is the set of inversions of the arrangement, where $\{i < j < k\} \in \binom{[n]}{3}$ is an *inversion* if the crossing of the pseudolines i and k occurs below j. The inversion set of the pseudoline arrangement in Figure 6.1 is $\{124, 134, 135, 234, 235\}$. Simple pseudoline arrangements also correspond to rhombic tilings of a zonogon via the Bohne-Dress Theorem [18],[85] as demonstrated in Figure 6.1.

A (non-simple) arrangement of pseudolines may have crossings involving more than two pseudolines. The set of simple arrangements that may be obtained by resolving these crossings forms a closed interval of $\mathrm{HB}(n,2)$, which we call a *facial interval*. For example, the arrangement in Figure 6.2 has two non-simple crossings that may be resolved in 16 ways, which is an interval of $\mathrm{HB}(6,2)$. One such resolution is the arrangement of Figure

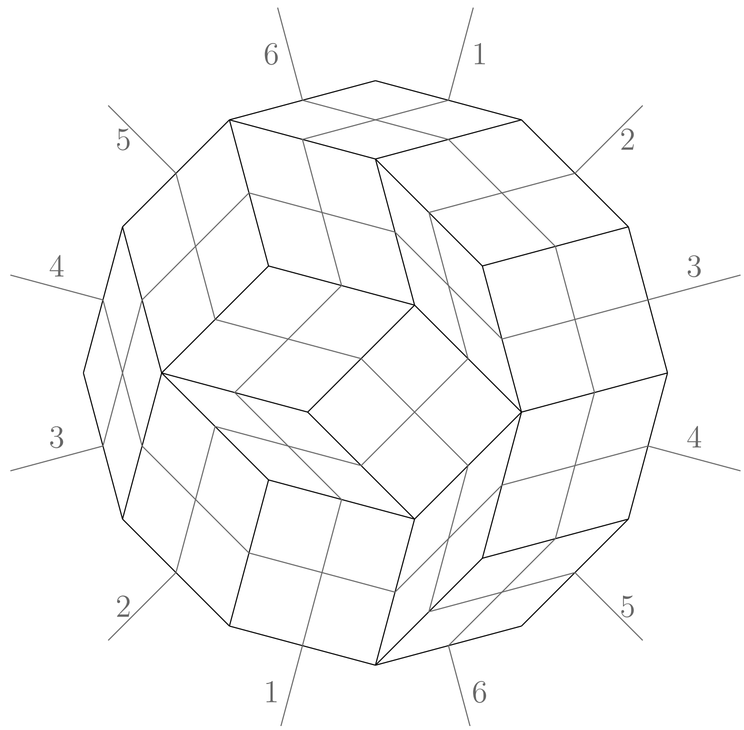


Figure 6.1: A rhombic tiling of a zonogon with its associated pseudoline arrangement.

6.1.

Rambau proved that that the proper part of $\mathrm{HB}(n, d)$ is homotopy equivalent to an $(n-d-2)$-sphere as an application of his Suspension Lemma [72]. Reiner extended this by showing that any facial interval of $\mathrm{HB}(n, d)$ is homotopy equivalent to a sphere [83, Conjecture 6.9]. He conjectured that every other interval is contractible. As $\mathrm{HB}(n, 1)$ is isomorphic to the weak order of the symmetric group on $[n]$, the conjectured homotopy type of intervals was already verified by Björner for $\mathrm{HB}(n, 1)$ [10]. We prove Reiner's conjecture for $\mathrm{HB}(n, 2)$.

Theorem 6.0.2 *An interval of* $\mathrm{HB}(n, 2)$ *is non-contractible if and only if it is facial.*

Björner's computation of the homotopy type of intervals of $\mathrm{HB}(n, 1)$ relies on the lattice property of the weak order. Although $\mathrm{HB}(n, 2)$ is not a lattice when $n \geq 6$, it is "close enough" to being a lattice that a similar argument may be applied. Recall that for any poset P, the order complex of P is homotopy equivalent to the order complex of P_{nonc} [99, Proposition 6.1] (Lemma 3.3.9). We prove that if P is any open interval of $\text{HB}(n, 2)$, then either P_{nonc} is the proper part of a Boolean lattice, or P_{nonc} contains an element X such that $X \vee Y$ exists in P_{nonc} for all $Y \in P_{\text{nonc}}$. The latter intervals are contractible by a join-contraction argument.

By Theorem 6.0.2, there is a poset isomorphism between the non-contractible intervals of $\text{HB}(n, 2)$ ordered by inclusion and the lifting space of a central arrangement of n lines, ordered by weak maps. Using some general techniques in poset topology, this isomorphism implies that the lifting space is homotopy equivalent to a sphere of dimension $n - 3$. More generally, the lifting space of an alternating matroid is known to be homotopy equivalent to a sphere ([94] Theorem 4.12; see Figure 1.7). Reiner's conjecture would provide an alternate proof of this result.

The sphericity of the lifting space of an alternating matroid is an instance of the Generalized Baues Problem of Billera, Kapranov, and Sturmfels [7] or the Extension Space Problem of Sturmfels and Ziegler [94]; see [83, Question 2.2,2.3]. The *Baues poset* associated to a strong map $\mathcal{M} \twoheadrightarrow \mathcal{N}$ of oriented matroids is the set of single-element liftings $(\mathcal{O}, g)$ of $\mathcal{N}$ for which $\mathcal{O} \setminus g$ is a strong map image of $\mathcal{M}$, ordered by weak maps. Alternatively, given a linear projection of polytopes $P \to Q$, one may define a Baues poset of tilings of Q by images of faces of P, ordered by refinement. The *Generalized Baues Problem* is to determine whether the proper part of a given Baues poset is homotopy equivalent to a sphere of dimension $\text{rk}\,\mathcal{M} - \text{rk}\,\mathcal{N} - 1$ (or $\dim P - \dim Q - 1$). The oriented matroid and polytopal formulations of the Generalized Baues Problem are equivalent on projections of zonotopes via the Bohne-Dress Theorem.

We would like to find other examples of strong maps $\mathcal{M} \twoheadrightarrow \mathcal{N}$ (or projections $P \to Q$) for which there is a poset structure on the generic liftings whose non-contractible intervals are parameterized by the Baues poset as in Theorem 6.0.2. Known cases of this correspondence include the poset of chambers of a hyperplane arrangement [36] and the Tamari lattice. Other posets for which this is conjectured but not known include the (other) higher Bruhat orders, the higher Stasheff-Tamari orders [83, Conjecture 6.9], and the poset of reduced galleries of a supersolvable hyperplane arrangement [66].

The rest of this section is organized as follows. We prove some basic results on general higher Bruhat orders in §6.1. Wiring diagrams are defined in §6.3 along with

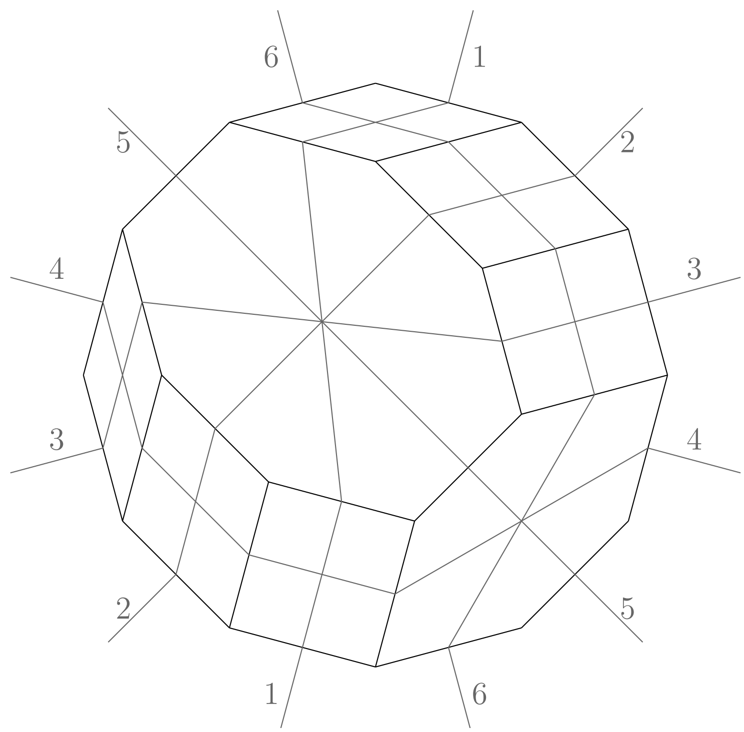


Figure 6.2: A zonogonal tiling with its associated non-simple pseudoline arrangement.

other results specific to the second higher Bruhat orders. Finally, the proof of Theorem 6.0.2 is given in §6.4.

6.1 Higher Bruhat orders

The Higher Bruhat orders may be defined in a variety of equivalent ways; see Theorem 6.1.1 for a partial list. We summarize the key definitions below.

Let $\binom{[n]}{d+1}$ denote the $(d+1)$-element subsets of $\{1,\ldots,n\}$. We say a subset X of $\binom{[n]}{d+1}$ is *closed* if $I \cup \{i,j\} \in X$ and $I \cup \{j,k\} \in X$ implies $I \cup \{i,k\} \in X$ for $I \in \binom{[n]}{d-1}$, $i,j,k \in [n] - I$, $i < j < k$. For instance, $\{123, 134\}$ is not a closed subset of $\binom{[4]}{3}$ since it contains $\{1\} \cup \{2,3\}$ and $\{1\} \cup \{3,4\}$ but not $\{1\} \cup \{2,4\}$. A subset X of $\binom{[n]}{d+1}$ is *consistent* (or *biclosed*) if both X and $\binom{[n]}{d+1} - X$ are closed.

A d-packet $\mathcal{P}$ is the set of d-subsets of a $(d+1)$-subset of $[n]$. We also refer to the $(d+1)$-subset itself as a d-packet when expedient. A permutation π of $\binom{[n]}{d}$ is *admissible* if the elements of any d-packet are either in lex or reverse lex order in π. The set of admissible permutations of $\binom{[n]}{d}$ is denoted $\mathrm{AP}(n,d)$. The set $\mathrm{Inv}(\pi)$ of *inversions* of an admissible permutation π is the collection of d-packets in reverse lex order in π.

A d-dimensional *cyclic zonotope* with n zones is the image of a standard n-cube under the linear map

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_n \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{k-1} & t_2^{k-1} & \ldots & t_n^{k-1} \end{pmatrix},$$

where $t_1 < \cdots < t_n$. The vector configuration $v_1, \ldots, v_n$ consisting of the columns of this matrix realizes the *alternating matroid*, so called because the coefficients in any minimal linear dependence relation alternate in sign. The arrangement of hyperplanes othogonal to this vector configuration is a *cyclic hyperplane arrangement*.

Roughly speaking, a *cubical tiling* of a d-dimensional cyclic zonotope is a collection of d-dimensional faces of an n-cube whose images under the above linear map properly tile the cyclic zonotope. A *single-element extension* of a cyclic hyperplane arrangement is the addition of a "pseudohyperplane" symmetric through the origin to the arrangement. For more precise definitions, we refer to the book [15].

Theorem 6.1.1 (Manin-Schechtman [64], Ziegler [102]) *Fix $1 \leq d \leq n$. There are natural bijections among the following collections.*

1. *Inversion sets of admissible permutations of $\binom{[n]}{d}$.*
2. *Consistent subsets of $\binom{[n]}{d+1}$, ordered by single step inclusion.*
3. *Generic single element extensions of a cyclic hyperplane arrangement with n hyperplanes in $\mathbb{R}^{n-d}$.*
4. *Cubical tilings of a cyclic zonotope in $\mathbb{R}^d$ with n zones.*

The equivalence of 6.1.1(3) and 6.1.1(4) is given by the Bohne-Dress Theorem [18].

Unlike the usual convention, consistent sets are typically ordered by *single-step inclusion*; that is $X \leq Y$ holds if there exists a sequence $X = X_0 \subseteq \cdots \subseteq X_t = Y$ of consistent sets for which $|X_i - X_{i-1}| = 1$ for all i. For example, $B(n, 1)$ may be identified with the weak order on the symmetric group on $[n]$. The second higher Bruhat order $B(n, 2)$ defines an ordering on the commutation-equivalence classes of reduced words for the longest element of the symmetric group on $[n]$.

Ziegler showed that $\mathrm{HB}(n, d)$ is not ordered by inclusion of inversion sets in general. The smallest counterexample is in $\mathrm{HB}(8, 3)$. Since this counterexample will be considered later, we state this as a theorem here.

Theorem 6.1.2 *(Ziegler)[102]* $\mathrm{HB}(n, d) = \mathrm{HB}_{\subseteq}(n, d)$ *if* $n - d \leq 4$. *However* $\mathrm{HB}(8, 3)$ *is not ordered by inclusion. In particular,* $U_1 \subset U_2$ *but* $U_1 \not\leq U_2$ *where*

$$U_1 = \{1234,\ 5678\}$$

$$U_2 = \binom{[8]}{4} \setminus \left\{ \begin{matrix} 1235, & 1245, & 1345, & 2345, & 1236, & 1246, & 1346, & 2346, & 1256, \\ 4678, & 4578, & 4568, & 4567, & 3678, & 3578, & 3568, & 3567, & 3478 \end{matrix} \right\}$$

We have already seen that $\mathrm{HB}(n, 1)$ is a lattice. Using Theorem 6.1.1(4), it can be shown that $\mathrm{HB}(n, d)$ is a lattice whenever $n - d \leq 3$. In fact, these are the only cases for which $\mathrm{HB}(n, d)$ is a lattice [102]. Nevertheless, we will compute the homotopy type of open intervals of $\mathrm{HB}(n, 2)$ by an argument involving join-contraction.

The *Higher Bruhat order* $\mathrm{HB}(n, d)$ is the poset of consistent subsets of $\binom{[n]}{d+1}$ ordered by *single-step inclusion*; that is, $X \leq Y$ if there exists a sequence of consistent subsets

$X_0 \subseteq \cdots \subseteq X_t$ such that $X = X_0,\ Y = X_t$ and $|X_i \setminus X_{i-1}| = 1$ for all i. The same set ordered by ordinary inclusion is denoted $\mathrm{HB}_{\subseteq}(n, d)$. The posets $\mathrm{HB}(n, d)$ and $\mathrm{HB}_{\subseteq}(n, d)$ are both graded with rank function $X \mapsto |X|$ ([102] Theorem 4.1(G)).

When $d = 1$, $\mathrm{HB}(n, 1)$ is isomorphic to the weak order on the symmetric group, and the two orders $\mathrm{HB}(n, 1)$ and $\mathrm{HB}_{\subseteq}(n, 1)$ coincide. The weak order $\mathrm{HB}(n, 1)$ is a lattice where the join of X and Y is $\overline{X \cup Y}$. If $d \geq 2$, the poset $\mathrm{HB}(n, d)$ may not be a lattice; in particular, $\mathrm{HB}(6, 2)$ is not a lattice ([102] Theorem 4.4). Ziegler proved that $\mathrm{HB}(n, d) = \mathrm{HB}_{\subseteq}(n, d)$ when $n - d \leq 4$, but $\mathrm{HB}(8, 3)$ is weaker than $\mathrm{HB}_{\subseteq}(8, 3)$ ([102] Theorem 4.5). In fact, his example in $\mathrm{HB}(8, 3)$ shows that $\overline{X \cup Y}$ need not be the join of $X, Y \in \mathrm{HB}(n, d)$ even if $\overline{X \cup Y}$ is consistent.

For $X, Y \in \mathrm{HB}(n, d)$, if $X \subseteq Y$ we define the *ascent set*

$$\mathrm{Asc}(X, Y) = \{I \in Y \setminus X :\ X \cup \{I\} \in \mathrm{HB}(n, d)\}.$$

If $Y = \hat{1}$, we write $\mathrm{Asc}(X)$ for $\mathrm{Asc}(X, Y)$.

Lemma 6.1.3 *Fix $X \in \mathrm{HB}(n, d)$. The ascent set $\mathrm{Asc}(X)$ decomposes as the disjoint union $\mathrm{Asc}(X) = A_1 \sqcup \cdots \sqcup A_N$ where*

1. *$A_t = \{\{a_{t1} < \cdots < a_{t,d+1}\}, \{a_{t2} < \cdots < a_{t,d+2}\}, \ldots, \{a_{t,r_t} < \cdots < a_{t,d+r_t}\}\}$ (i.e. A_t is the set of contiguous intervals in the set $\{a_{t1}, \ldots, a_{t,r_t+d}\} \subseteq [n]$), and*

2. *if $I \in A_s,\ J \in A_t,\ s \neq t$ then $|I \cap J| < d$.*

Proof: We first show that any ascent $I \in \mathrm{Asc}(X)$ shares d elements with at most two other ascents of X. Suppose $I, J \in \mathrm{Asc}(X)$ such that $|I \cap J| = d$ with $I < J$ in lexicographic order. The restriction $X|_{I \cup J}$ is an element of $\mathrm{HB}(|I \cup J|, d)$ with two ascents, so it must be the bottom element. Consequently, the I (J) is the lex-minimal (lex-maximal) $(d + 1)$-subset of $I \cup J$.

Now suppose $J' \in \mathrm{Asc}(X),\ J' \neq J$ such that $J' > I$ in lexicographic order and $|J' \cap I| = d$. Then J' is the lex-maximal $(d+1)$-subset of $I \cup J'$ by the above argument. But, $|J \cap J'| = d$ and J, J' are not at opposite ends of their $(d+1)$-packet, a contradiction.

We have now established that for any $I \in \mathrm{Asc}(X)$, there is at most one $J > I$ in lexicographic order for which $|I \cap J| = d$. By similar reasoning, there is at most one

$L < I$ with $|I \cap L| = d$. Thus, $\mathrm{Asc}(X)$ decomposes into chains $I^t_1 < I^t_2 < \cdots < I^t_{m_t}$ where $|I^t_i \cap I^t_{i+1}| = d$ and all other intersections have cardinality strictly less than d. ∎

Lemma 6.1.4 *If $X \in \mathrm{HB}(n,d)$, then $X \cap \overline{\mathrm{Asc}(X)} = \emptyset$.*

Proof: The ascent set $\mathrm{Asc}(X)$ is a subset of $\binom{[n]}{d} \setminus X$, and the latter set is closed. Hence, $X \cap \overline{\mathrm{Asc}(X)}$ is empty. ∎

6.2 Connectivity

The connectivity of the graph of cubical tilings of a cyclic zonotope is an important feature. In this section, we consider a couple refinements of this connectivity result.

6.2.1 Simple connectivity

The interpretation of $\mathrm{HB}(n,d)$ as maximal chains in $\mathrm{HB}(n,d-1)$ modulo diamonds has an interesting topological consequence for the graphs of these posets. Let $\mathbf{X}^{(1)}_{n,d}$ denote the (undirected) Hasse diagram of $\mathrm{HB}(n,d)$ and let $\mathbf{X}^{(2)}_{n,d}$ be the 2-dimensional regular CW-complex whose 1-skeleton is $\mathbf{X}^{(1)}_{n,d}$ with a 2-cell attached to every diamond and $(2d+4)$-circuit as in Figure 6.3.

Proposition 6.2.1 $\mathbf{X}^{(2)}_{n,d}$ *is simply connected.*

Proof: We deduce the simple connectivity of $\mathbf{X}^{(2)}_{n,d}$ from the connectivity of $\mathbf{X}^{(1)}_{n,d+1}$. Let γ be some closed path in $\mathbf{X}^{(1)}_{n,d}$. We show γ is nullhomotopic by an induction on the rank and cardinality of the highest rank elements of $\mathrm{HB}(n,d)$ visited by γ.

If γ only visits a single vertex of the graph $\mathbf{X}^{(1)}_{n,d}$, we are done.

Let $x \in \mathrm{HB}(n,d)$ be some element of maximal rank in the path of γ. Let y, z be the immediate neighbors of some occurence of x in γ. By assumption, $\mathrm{rk}(y) = \mathrm{rk}(z) = \mathrm{rk}(x) - 1$. Choose saturated chains m_y, m_z in the intervals $[\hat{0}, y]$, $[\hat{0}, z]$ and fix some saturated chain m in $[x, \hat{1}]$. Let f be the concatenation of chains $m_y \circ (y - x) \circ m$ and g the concatenation $m_z \circ (z - x) \circ m$. Since the space $A(n, d+1)$ of maximal chains is

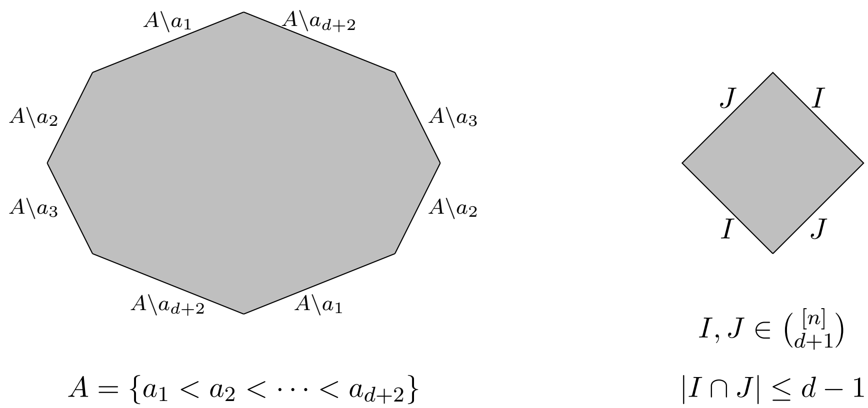


Figure 6.3: $\mathbf{X}^{(2)}_{n,d}$ is constructed by attaching 2-cells to certain cycles in the graph of $\mathrm{HB}(n, d)$.

connected by diamond and $(2d + 4)$-cycle flips, f and g are homotopic in $\mathbf{X}^{(2)}_{n,d}$. Basing the loop $\overline{f} \circ g$ at z and concatenating with γ replaces the path $y - x - z$ with $\overline{m_1} \circ m_2$. This yields a path homotopic to γ either with lower maximal rank or fewer elements of maximal rank $\mathrm{rk}(x)$. ∎

When $d = 2$, this proposition was proved by Shapiro, Shapiro, and Vainshtein in a very similar manner [90]. The proof essentially showed that the simple connectivity of $\mathbf{X}^{(2)}_{n,d}$ follows from the connectivity of $\mathbf{X}^{(1)}_{n,d+1}$. In principle, it should be possible to generate the higher order syzygies of these complexes by iterating this method.

6.2.2 Local connectivity

The vertex sets that can appear in a rhombic tiling were characterized by Leclerc and Zelevinsky in [60] as maximal collections of strongly separated sets. Two sets $I, J \subseteq [n]$ are *strongly separated* if either $I - J \prec J - I$ or $J - I \prec I - J$ where $A \prec B$ means $a < b$ for all $a \in A,\ b \in B$. A collection of subsets $\mathcal{C}$ of $[n]$ is strongly separated if I, J are strongly separated for all $I, J \in \mathcal{C}$.

Let $\Gamma(n) \subseteq 2^{[n]}$ be the (flag) simplicial complex of strongly separated families. Leclerc and Zelevinsky showed that this complex is pure and its facets are the vertex sets of rhombic tilings. Another way to state this result is that the collections of rhombic tilings containing a particular vertex satisfy a Helly property: If $v_i \subseteq [n]_{i \in I}$ and

$X(v_i)$ is the set of rhombic tilings containing v_i, then $X(v_i) \cap X(v_j) \neq \emptyset$ $(\forall i, j)$ implies $\bigcap_i X(v_i) \neq \emptyset$.

Henriques and Speyer showed that $X(v_i)$ is a connected subgraph of the graph of tilings [46]. We give an alternate proof in terms of Coxeter groups.

Proposition 6.2.2 *$X(A)$ is connected for all $A \subseteq [n]$.*

Proof: We convert the proposition into the language of Coxeter groups and apply connectivity of graphs of reduced words.

$X(A)$ is the full graph of zonogon tilings if $A = \emptyset$ or $A = [n]$, so we may assume A is neither of these.

The graph of zonogon tilings is a contraction of the graph $\mathcal{G}(w_0)$ of reduced words for w_0 where each tiling corresponds to a commutation class of words. Thus, if the inverse image $\widetilde{X}(A)$ is connected, then so is $X(A)$. The vertices of the n-cube (except $\emptyset, [n]$) naturally biject with cosets $uW_{\hat{j}}$ of maximal parabolic subgroups of $\mathfrak{S}_n$. The bijection takes a subset A and maps it to those permutations $w \in \mathfrak{S}_n$ for which $w^{-1}([|A|]) = A$, i.e. w starts with a permutation of A and ends with a permutation of $[n] \setminus A$. If $A \subsetneq [n]$ corresponds to uW_I under this bijection, then

$$\widetilde{X}(A) = \{s_{i_1} \cdots s_{i_N} \in \mathcal{G}(w_0) \mid (\exists l)\ s_{i_1} \cdots s_{i_l} \in uW_I\}.$$

Let $s_{i_1} \cdots s_{i_N} \in \widetilde{X}(A)$. Elements $w \in uW_I$ have a unique decomposition $w = w^I w_I$ where $l(w) = l(w^I) + l(w_I)$ and $w_I \in W_I$. Since the graph of reduced words of w is connected, we may transform $s_{i_1} \cdots s_{i_l}$ into some other word $s_{j_1} \cdots s_{j_l}$ by a sequence of braid moves, where $s_{j_1} \cdots s_{j_m} = w^I$ for some $m \leq l$. The braid moves preserve the product, so this produces a sequence of reduced words for w_0

$$s_{i_1} \cdots s_{i_l} \cdots s_{i_N} \sim \cdots \sim s_{j_1} \cdots s_{j_l} s_{i_{l+1}} \cdots s_{i_N},$$

where every term in the sequence lives in $\widetilde{X}(A)$. By a similar argument, we may transform $s_{j_1} \cdots s_{j_m}$ into some canonical form for w^I and $s_{j_{m+1}} \cdots s_{i_N}$ into a canonical form for $(w^I)^{-1} w_0$ via braid moves without leaving $\widetilde{X}(A)$. By transforming $s_{i_1} \cdots s_{i_N}$ into a canonical factorization, we have demonstrated the connectivity of $\widetilde{X}(A)$. ∎

6.3 The second Higher Bruhat order

A *wiring diagram* is a collection of *wires*, continuous piecewise linear curves $C_1, \ldots, C_n$ in $\mathbb{R}^2$, satisfying the following conditions.

1. The projection of C_i onto the first coordinate is bijective.

2. The wires are in order $C_1, \ldots, C_n$ top-to-bottom, sufficiently far to the right.

3. Distinct wires C_i, C_j cross at a unique point.

4. All crossings are transverse.

We shall further assume that the wiring diagram is *simple*, meaning there are no common intersections among three or more wires. In particular, each wire C_i determines a permutation $\pi_i = a_1 \cdots a_{n-1}$ of $[n] \backslash i$ where if $r < s$ then the first coordinate of $C_i \cap C_{a_r}$ is less than that of $C_i \cap C_{a_s}$. Two wiring diagrams are considered equivalent if they determine the same sequence of wire permutations $(\pi_i)_{i \in [n]}$.

For $1 \leq i < j < k \leq n$, if the crossing of C_i and C_k is below (above) C_j, then $\{i, j, k\}$ is an *inversion triple* (*non-inversion triple*). The map taking a wiring diagram to its set of inversion triples defines a bijection between equivalence classes of simple wiring diagrams with n wires and consistent subsets of $\binom{[n]}{3}$.

Let $x, y \in \mathrm{HB}(n, 2)$ with $x \subseteq y$. A *difference triple* is an element of $y - x$. A *difference block* is a subset of $y - x$ of the form $\{\{i_j, i_{j+1}, i_{j+2}\} : \ j \in [m]\}$, where $1 \leq i_1 < \cdots < i_{m+2} \leq n$.

For distinct $i, j, k \in [n]$, the piece S^j_{ik} of C_j between $C_i \cap C_j$ and $C_k \cap C_j$ is called a *segment* of C_j. If $\{i, j, k\}$ is a non-inversion triple, $i < j < k$, then the *floor* of $\{i, j, k\}$ is the segment S^j_{ik}. The floor is *elementary* if its interior is not intersected by any other wire (i.e. i and k are adjacent in π_j). The *height* of a non-inversion triple $\{i, j, k\}$ is the number of wires that pass below the segment S^j_{ik}.

Proposition 6.3.1 ([38]) *Let W be a simple wiring diagram with inversion set x. Let $y \in \mathrm{HB}(n, 2)$ such that $x \subsetneq y$.*

1. *There exists an element of $y - x$ with an elementary floor in W. ([38] Lemma 2.2)*

2. *Among those elements of $y-x$ with an elementary floor, if I is of maximum height, then $x\cup\{I\}$ is consistent. In particular,* $\mathrm{Asc}(x,y)$ *is nonempty. ([38] Lemma 2.3)*

Corollary 6.3.2 *The second higher Bruhat order* $\mathrm{HB}(n,2)$ *is ordered by inclusion; that is,* $\mathrm{HB}(n,2) = \mathrm{HB}_{\subseteq}(n,2)$ *as posets.*

Given $\mathcal{I} \subseteq \binom{[n]}{d+1}$, let $[\mathcal{I}]$ denote the union $\bigcup_{I\in\mathcal{I}} I$.

Lemma 6.3.3 *Let $x \in \mathrm{HB}(n,2)$ have wiring diagram W. If $\mathcal{I}$ is a block with an elementary floor in W, then $x \cup \overline{\mathcal{I}}$ is not consistent if and only if there exists a wire p intersecting the segments $S^{i_0}_{i_1,i_m}$ and $S^{i_m}_{i_0,i_{m-1}}$ where $[\mathcal{I}] = \{i_0 < \cdots < i_m\}$.*

Proof: Let $\mathcal{I}$ be a block with an elementary floor in W, and let $p \in [n]-[\mathcal{I}]$. If $x\cup\{\mathcal{I}\}$ is consistent, then for $i \in [\mathcal{I}]$ the words π_i have the elements of $[\mathcal{I}]\setminus i$ flipped with the other letters in the same relative order. Hence, $x\cup\{\mathcal{I}\}$ is consistent if and only if every wire in $[n]-[\mathcal{I}]$ does not intersect any segment S^k_{ij} for $i,j,k \in [\mathcal{I}]$.

If $x\cup\{\mathcal{I}\}$ is not consistent, then there exists a wire p intersecting some segment S^k_{ij} for $i,j,k\in[\mathcal{I}]$. As $\mathcal{I}$ has an elementary floor in W, p must intersect the segments $S^{i_0}_{i_1,i_m}$ and $S^{i_m}_{i_0,i_{m-1}}$ by planarity. ∎

To determine the homotopy type of intervals of $\mathrm{HB}(n,2)$, we use a stronger version of Proposition 6.3.1(2).

Proposition 6.3.4 *Let W be a simple wiring diagram with inversion set x. Let $y \in \mathrm{HB}(n,2)$ such that $x \subseteq y$. Among the difference blocks of $y-x$ with an elementary floor, if $\mathcal{I}$ is of maximum height, then $x\cup\overline{\mathcal{I}}$ is consistent.*

Proof: Let $\mathcal{I}$ be a difference block of $y-x$ with an elementary floor, and assume $x\cup\overline{\mathcal{I}}$ is not consistent. Replacing $\mathcal{I}$ by a smaller block, we may assume that $x\cup\overline{\mathcal{I}'}$ is consistent for every block $\mathcal{I}'$ that is a proper subset of $\mathcal{I}$. Let $[\mathcal{I}] = \{i_0,\ldots,i_m\}$ where $i_0<\cdots<i_m$. By Lemma 6.3.3, there exists a wire p intersecting the segments $S^{i_0}_{i_1,i_m}$ and $S^{i_m}_{i_0,i_{m-1}}$. By the minimality of $\mathcal{I}$, every such wire intersects the subsegments $S^{i_0}_{i_{m-1},i_m}$ and $S^{i_m}_{i_0,i_1}$; see Figure 6.4.

Let P be the set of wires intersecting $S^{i_0}_{i_{m-1},i_m}$ and $S^{i_m}_{i_0,i_1}$. If $p\in P$, we claim that $i_0<p<i_m$ and $\{i_0,p,i_m\}$ is a difference triple in $y-x$. This follows by restriction of W to the wires $\{i_0,p,i_1,i_m\}$.

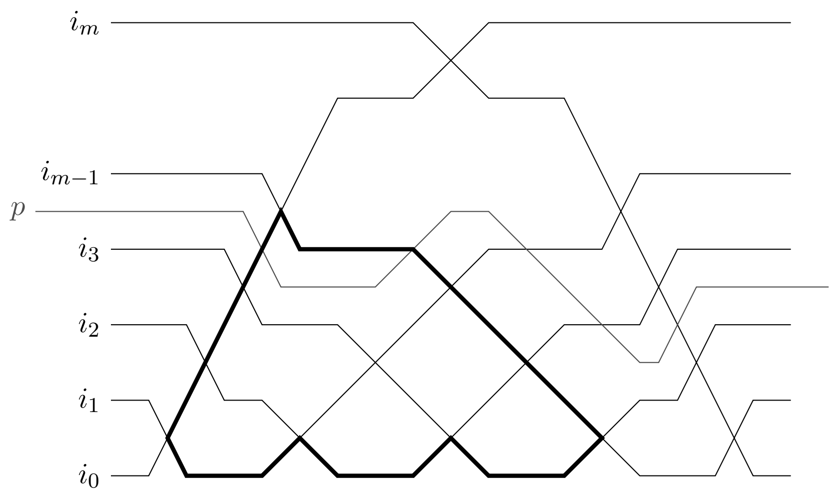


Figure 6.4: Wire p intersects $S^{i_0}_{i_1 i_{m-1}}$ so $x \cup \overline{\mathcal{I} \setminus \{i_{m-2}, i_{m-1}, i_m\}}$ is not consistent.

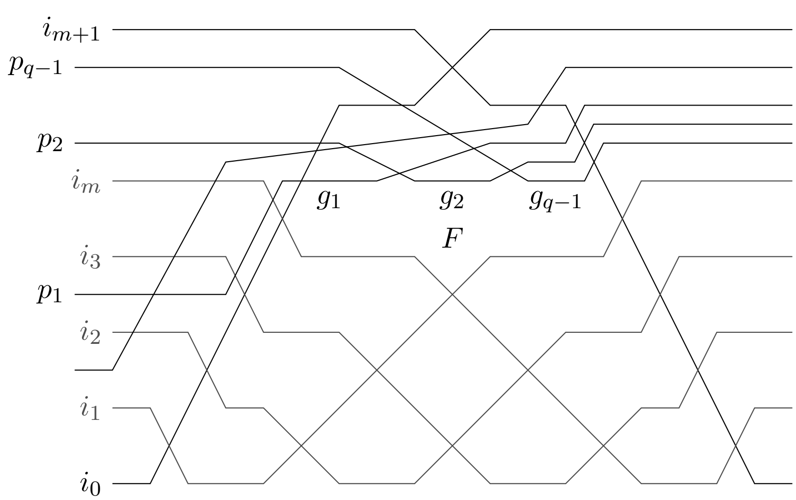


Figure 6.5: Proof of Proposition 6.3.4.

Let F denote the region above the wires $i_1, \cdots, i_{m-1}$, below i_0, i_{m+1} and below all of the wires in P. As shown in Figure 6.5, we label the upper edges of F by $g_0, g_1, \ldots, g_q$ which are supported by the wires $i_0 = p_0 < p_1 < \cdots < p_{q-1} < p_q = i_m$.

We show by induction that one of the $g_j,\ j \in [q-1]$ is the floor of a difference triple in $y \setminus x$. We are given that $\{i_0, p_j, i_m\}$ is in y. Suppose $p_{j-1}p_j i_m \in y$. Using the packet $\{p_{j-1}, p_j, p_{j+1}, i_{m+1}\}$ either $p_{j-1}p_jp_{j+1} \in y$ or $p_jp_{j+1}i_m \in y$. The former case has an elementary floor g_j. Induction on j completes the argument.

Hence there exists a difference triple $\{p_{j-1}, p_j, p_{j+1}\}$ with an elementary floor. By Proposition 6.3.1(2), there exists a difference triple I for which $x \cup \{I\}$ is consistent whose height is strictly greater than that of $\mathcal{I}$.

■

Lemma 6.3.5 *Let* $x \in \mathrm{HB}(n, 2)$ *and* $A(x) = A_1 \sqcup \cdots \sqcup A_N$ *as in Lemma 6.1.3. For all* $s \neq t$*, if* $|[A_s] \cap [A_t]| \geq 2$ *and* $\mathrm{ht}(A_s) \leq \mathrm{ht}(A_t)$ *then*

$$[A_s] \cap [A_t] = \{\min[A_t], \max[A_t]\}.$$

Proof: Let s, t be distinct indices with $|[A_s] \cap [A_t]| \geq 2$ and $\mathrm{ht}(A_s) \leq \mathrm{ht}(A_t)$. Let $i, k \in [A_s] \cap [A_t]$ such that $i < k$. We let W denote a wiring diagram of x.

We first show that $\mathrm{ht}(A_s) < \mathrm{ht}(A_t)$. By Lemma 6.1.3(2) there exists $q \in [A_t] \setminus [A_s]$ such that $i < q < k$. Let $[i, k] \cap [A_s] = \{i = j_0 < j_1 < \cdots < j_r < j_{r+1} = k\}$ and let e_α be the base of $j_{\alpha-1}j_\alpha j_{\alpha+1}$ for $1 \leq \alpha \leq r$. Since $j_{\alpha-1}j_\alpha j_{\alpha+1}$ is an ascent, each e_α is a segment. Let $e = \bigcup_\alpha e_\alpha$ be the union of these segments. Then q does not intersect e_α. If q is above e_α, then $\mathrm{ht}(A_s) < \mathrm{ht}(iqk) = \mathrm{ht}(A_t)$ as desired. If q is below e_α, then $\mathrm{ht}(A_s) > \mathrm{ht}(iql) = \mathrm{ht}(A_t)$, contrary to the hypothesis.

Let $p \in [A_t] \setminus \{i, k\}$. It remains to show that $i < p < k$. From this, it follows that $[A_s] \cap [A_t]$ must intersect only at the 2 elements which lie at opposite ends of $[A_t]$.

Suppose to the contrary that $p < i$. Since $pil \notin x$, $\pi_p(i) < \pi_p(l)$. If $\min[A_s] < i$ then the base of pil includes the base of an ascent I in A_s. By assumption on the height of A_t, this implies $I \in A_t$, a contradiction. If $\min[A_s] = i$ then the base of pil includes the crossing $i \cap j$ where $j = \min([A_s] \setminus i)$. Consequently, $\mathrm{ht}(A_t) \leq \mathrm{ht}(A_s)$, a contradiction.

A symmetric argument shows that $p \not> l$, thus completing the proof. ■

The following proposition is the key to the proof of Theorem 6.0.2, as described in the introduction.

Proposition 6.3.6 *Let W be a simple wiring diagram with inversion set x. Let $y \in \mathrm{HB}(n,2)$ such that $x \subsetneq y$, and let $\mathcal{I} \subseteq \mathrm{Asc}(x,y)$ such that $[x, x \cup \overline{\mathcal{I}}]$ is facial. If $I_0 \in \mathrm{Asc}(x,y)$ is of maximum height in W, then $[x, x \cup \overline{\mathcal{I} \cup \{I_0\}}]$ is facial.*

Proof: (Reduce to checking that for the block $\mathcal{I}$ containing I_0, $[x, x \cup \overline{\mathcal{I}}]$ is facial.)

1: Since no two subsets $I \in \overline{\mathcal{I}_s}$, $J \in \overline{\mathcal{I}_t}$ appear in a common packet, to check the consistency of $x \cup \overline{\mathcal{I}_s \sqcup \mathcal{I}_t}$ it suffices to check $x \cup \overline{\mathcal{I}_s}$ and $x \cup \overline{\mathcal{I}_t}$ independently.

2: We proceed by induction on $|\mathcal{I} \backslash \mathcal{I}'|$. Suppose $x \cup \overline{\mathcal{I}}$ is consistent and let $\mathcal{I}' = \mathcal{I} \backslash \{J\}$. If J appears in block $\mathcal{I}_t$, then this block splits into

$$\mathcal{I}_t = \mathcal{I}'_{t,0} \sqcup \{J\} \sqcup \mathcal{I}'_{t,1}$$

and all other blocks remain the same. It suffices to show that $x \cup \overline{\mathcal{I}'}_{t,0} \cup \overline{\mathcal{I}'}_{t,1}$ is consistent.

Every packet $\mathcal{P}$ is either contained in $\overline{\mathcal{I}'}$, is disjoint from $\overline{\mathcal{I}'}$, or meets $\overline{\mathcal{I}'}$ in a single $(k+1)$-set I. Suppose we are in the last case. If $\mathcal{P} \cap \overline{\mathcal{I}} = \{I\}$, then we are done since by assumption $\mathcal{P} \cap (x \cup \overline{\mathcal{I}})$ is either initial or terminal in $\mathcal{P}$.

Assume $\mathcal{P} \cap \overline{\mathcal{I}} = \mathcal{P}$ and set $\{p\} = [\mathcal{P}] \setminus I$. Then $I \in \overline{\mathcal{I}'}_{t,0}$ or $\overline{\mathcal{I}'}_{t,1}$ and $p \in [\{J\} \cup \mathcal{I}'_{t,1}]$ or $p \in [\{J\} \cup \mathcal{I}'_{t,0}]$. In the first case, I is lex minimal in $\mathcal{P}$ while in the latter case, I is lex maximal. This proves.

3: Let $\mathrm{Asc}(x,y) = A_1 \sqcup A_N$ as in Lemma 6.1.3 and assume $I_0 \in A_1$. Let $\mathcal{I} \subseteq \mathrm{Asc}(x,y)$ such that $x \cup \overline{\mathcal{I}}$ is consistent. By Proposition 6.3.4, $x \cup \overline{A_1}$ is consistent. Then $x \cup \overline{\mathcal{I} - A_1}$ is consistent. $x \cup \overline{A_1 \cup \mathcal{I}}$ is consistent. Finally, $x \cup \overline{I_0 \cup \mathcal{I}}$ is consistent.

∎

6.4 Proof of Theorem 6.0.2

Theorem 6.4.1 *Let $x, y \in \mathrm{HB}(n,2)$ such that $x < y$. If $[x,y]$ is facial, then (x,y) is homotopy equivalent to a $(|\mathrm{Asc}(x,y)| - 2)$-sphere. Otherwise, (x,y) is contractible.*

Proof: Assume the statement holds for intervals (x, z) with $x \leq z < y$. By Lemma 3.3.9,

$$(x, y) \simeq (x, y)_{nonc}$$

where $(x, y)_{nonc} = \{z \in (x, y) \mid (x, z) \text{ is not contractible}\}$. Since $\mathrm{HB}(n, 2)$ is ordered by inclusion, $x \cup \overline{\mathcal{I}} \leq y$ whenever $\mathcal{I}$ is a subset of $\mathrm{Asc}(x, y)$ such that $x \cup \overline{\mathcal{I}}$ is consistent. By the inductive hypothesis,

$$(x, y)_{nonc} = \{\mathcal{I} \subseteq \mathrm{Asc}(x, y) : \ x \cup \overline{\mathcal{I}} \text{ is consistent}, \ x \cup \overline{\mathcal{I}} \neq y\}.$$

Suppose $y = x \cup \overline{\mathrm{Asc}(x, y)}$. By Proposition 6.3.6, $x \cup \overline{\mathcal{I}}$ is consistent for every subset $\mathcal{I}$ of $\mathrm{Asc}(x, y)$. Therefore,

$$(x, y)_{nonc} \cong (2^{\mathrm{Asc}(x,y)})_{\mathrm{prop}} \overset{homeo}{\cong} \mathbb{S}^{|\,\mathrm{Asc}(x,y)|-2}.$$

Now assume that either $y \neq x \cup \overline{\mathrm{Asc}(x, y)}$ holds or $x \cup \overline{\mathrm{Asc}(x, y)}$ is not consistent. From Proposition 6.3.6, $(x, y)_{nonc}$ is the face poset of a simplicial complex over $\mathrm{Asc}(x, y)$. By Proposition 6.3.6, this simplicial complex has a cone point $I_0 \in \mathrm{Asc}(x, y)$, so $(x, y)_{nonc}$ is contractible. ∎

Chapter 7

Gallery posets

Recall that a *reduced gallery* of a real central arrangement $\mathcal{A}$ is a sequence of chambers $c_0, c_1, \ldots, c_m$ such that adjacent chambers are separated by exactly one hyperplane, and c_0 and c_m are separated by m hyperplanes. For any codimension 2 subspace $X \in L(\mathcal{A})$, a gallery between opposite chambers $c_0, -c_0$ can cross the hyperplanes containing X in two ways. We introduce a poset $\mathrm{Gal}(\mathcal{A}, r_0)$ of reduced galleries r ordered by single-step inclusion of sets of codimension 2 subspaces separating r from r_0. The Hasse diagram of this poset is the usual graph of reduced galleries. An example is shown in Figure 7.1

In this chapter, we analyze the local topology of $\mathrm{Gal}(\mathcal{A}, r_0)$ when $\mathcal{A}$ is supersolvable and r_0 is incident to a modular flag. In this situation, we prove $\overline{\mathrm{Gal}}(\mathcal{A}, r_0)$ is homotopy equivalent to a $(\mathrm{rk}\,\mathcal{A} - 3)$-sphere using Rambau's Suspension Lemma. Similarly, we identify certain intervals, which we call *facial*, that are also homotopy equivalent to spheres. We conjecture that all other intervals are contractible.

Given a fundamental chamber c_0, a *cellular string* is a sequence of faces $(x_1, x_2, \ldots, x_m)$ in $\mathcal{L}_{\geq 1}(\mathcal{A})$ such that $x_1 \circ c_0 = c_0,\ x_m \circ (-c_0) = -c_0$ and

$$x_i \circ (-c_0) = x_{i+1} \circ c_0 \qquad (\forall i).$$

The poset $\omega(\mathcal{A}, c_0)$ of cellular strings is ordered by refinement: given cellular strings $\underline{x} = (x_1, \ldots, x_m),\ \underline{x}' = (x'_1, \ldots, x'_l)$, we say $\underline{x} \leq \underline{x}'$ if every x'_i is a face of some x_j. A maximally refined cellular string has all faces of codimension 1. Such cellular strings may be identified with reduced galleries. Hence, we say a gallery r is *incident to a*

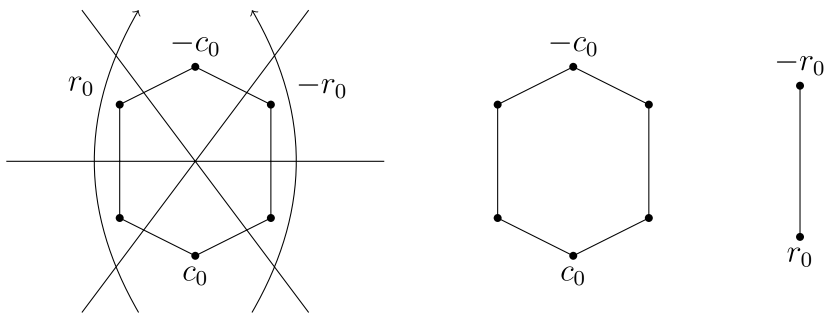


Figure 7.1: (Left) An arrangement $\mathcal{A}$ of three lines in $\mathbb{R}^2$. (Center) The poset of chambers of $\mathcal{A}$ with base chamber c_0. (Right) The poset of galleries with base gallery r_0.

cellular string $\underline{x}$ if $\underline{x} \leq r$ in the order on cellular strings.

Our main result is the following.

Theorem 7.0.2 *Let $\mathcal{A}$ be a real supersolvable hyperplane arrangement with chamber c_0 and gallery r_0 both incident to a modular flag. Let $\underline{x} = (x_1, x_2, \ldots, x_m) \in \omega(\mathcal{A}, c_0)$ be a cellular string of $\mathcal{A}$. The set of galleries incident to $\underline{x}$ forms a closed interval of* $\mathrm{Gal}(\mathcal{A}, r_0)$ *whose proper part is homotopy equivalent to* $\mathbb{S}^{\sum_{i=1}^{m} \mathrm{codim}(x_i) - m - 2}$.

We conjecture that all other intervals are contractible.

The motivation for this work primarily comes from two sources: the parameterization of noncontractible intervals in the chamber poset of an arrangement $\mathcal{A}$ proved by Edelman and Walker [36], and a conjecture by Reiner on the noncontractible intervals of the Higher Bruhat orders [83] (see Chapter 6). We hope to derive new instances of the generalized Baues problem posed by Billera, Kapranov, and Sturmfels via these posets [7].

The rest of the section is structured as follows. The above theorems are interpreted for Coxeter groups in §7.1. In §7.2, we use the Suspension Lemma (Lemma 3.3.6) to compute the homotopy type of the proper part of Gal. Theorem 7.0.2 is proved in §7.3.

7.1 Coxeter Groups

Recall from §2.4 that the faces of a reflection arrangement of a Coxeter system (W, S) are in natural bijection with parabolic cosets $\{wW_J : \ w \in W, \ J \subseteq S\}$. For $J \subseteq S$, let

$w_0(J)$ denote the longest element of the subsystem (W_J, J). Then a cellular string may be identified as a word $(J_1, \ldots, J_m)$ where $\emptyset \neq J_i \subseteq S$ for all i and

$$w_0(S) = w_0(J_1)w_0(J_2)\cdots w_0(J_m), \qquad l(w_0(S)) = \sum_{i=1}^{m} l(w_0(J_i)).$$

The cellular strings are ordered by refinement, i.e. $(I_1, \ldots, I_l) \leq (J_1, \ldots, J_m)$ if there is are indices $0 = \alpha_1 < \cdots < \alpha_t < l$ such that

$$w_0(J_k) = w_0(I_{\alpha_k+1})w_0(I_{\alpha_k+2})\cdots w_0(I_{\alpha_{k+1}}) \qquad l(w_0(J_k)) = \sum_{i=1}^{\alpha_{k+1}-\alpha_k} l(w_0(I_{\alpha_k+i})).$$

In this language, Theorem 7.0.2 may be restated as follows.

Theorem 7.1.1 *Let (W, S) be a Coxeter system of type A or B. Let r_0 be a reduced word incident to a modular flag of the corresponding reflection arrangement. The set of reduced words for w_0 refining a given cellular string $(J_1, \ldots, J_m)$ forms an interval $[r, r']$ in $\mathrm{Gal}(\mathcal{A}, r_0)$ such that (r, r') is homotopy equivalent to $\mathbb{S}^{\sum_i |J_i| - m - 2}$.*

We conjecture that all other intervals are contractible. This theorem and conjecture are both highly dependent on the choice of reduced word r_0. Other choices for r_0 yield non-isomorphic posets which fail to have the nice local topology. For type A, the base word is

$$r_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots,$$

while in type B, the base is

$$r_0 = s_0(s_1s_0s_1)(s_2s_1s_0s_1s_2)\cdots.$$

At this time it is unclear whether a good choice of r_0 exists for every finite Coxeter group.

7.2 Homotopy type of the gallery poset

The second part of the proof of Theorem 4.3.4 proves the following proposition.

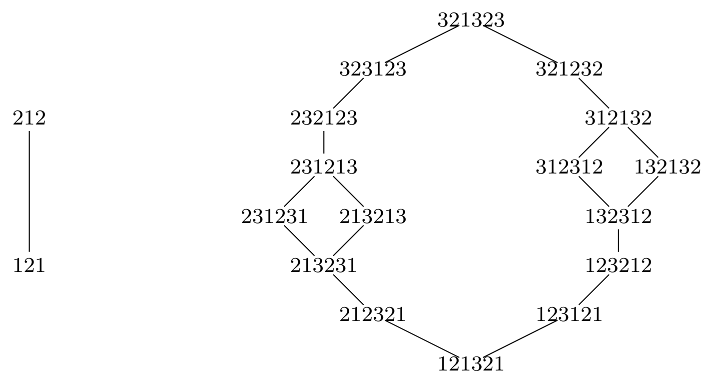


Figure 7.2: (Left) $\mathrm{Gal}(\mathcal{A}, r_0)$ for the type A_2 Coxeter system. (Center) $\mathrm{Gal}(\mathcal{A}, r_0)$ for type A_3.

Proposition 7.2.1 *(RR [81]) Assume that l is a modular ray and let c_0 be a chamber incident to l. Let π denote the function $r \mapsto r_l$ sending galleries of $\mathcal{A}$ from c_0 to $-c_0$ to galleries of $\mathcal{A}_l$. Given a reduced gallery $r_l \in \mathcal{A}_l$, let r_1 be the unique gallery incident to l in the fiber $\pi^{-1}(r_l)$ defined by Proposition 4.3.1(2). If $r_1 \neq r_2 \in \pi^{-1}(r_l)$ there exists $X \in L_2(r_1, r_2)$ incident to r_2.*

Theorem 7.2.2 *Let r_0 be a gallery incident to a modular flag of a rank d supersolvable arrangement $\mathcal{A}$. The proper part of the poset of reduced galleries $\mathrm{Gal}(\mathcal{A}, r_0)$ is homotopy equivalent to a $(d-3)$-sphere.*

Proof: We begin by defining the maps and objects in the Suspension Lemma 3.3.6. Let $\mathcal{F}$ be a modular flag of faces

$$\mathcal{F}: \; c_0 = F_0 < F_1 < \cdots < F_d = 0, \; F_i \in \mathcal{L}(\mathcal{A}),$$

and set $l = F_{d-1}$. Let r_0 be the unique gallery incident to $\mathcal{F}$.

Define gallery posets $P = \mathrm{Gal}(\mathcal{A}, r_0)$, $Q = \mathrm{Gal}(\mathcal{A}_l, (r_0)_l)$, and let $f : P \to Q$ be usual localization map removing hyperplanes not containing l.

Define a section $i : Q \to P$ by lifting a gallery r_l in Q along l to a gallery from $l \circ c_0$ to $l \circ (-c_0)$ and completing it in the unique way described in Proposition 2. We similarly define a section $j : Q \to P$ by lifting along $-l$.

Let (H, H') be the unique pair of adjacent hyperplanes in r_0 such that $H \supseteq l$ and $H' \not\supseteq l$. We claim that $X_0 = H \cap H'$ is the unique codimension 2 subspace incident to r_0 not containing l. Uniqueness holds since any $Y \in L_2(\mathcal{A}) \setminus L_2(\mathcal{A}_l)$ contains some hyperplane of $\mathcal{A}_l$ by the modularity of l, but the hyperplanes in $\mathcal{A}_l$ appear in r_0 before those of $\mathcal{A} \setminus \mathcal{A}_l$. Incidence at X_0 follows from Proposition 4.3.1(3) and Corollary 4.3.3. Let J denote the order ideal $\{r \in P \mid X_0 \notin L_2(r_0, r)\}$.

With this setup, we verify the three properties in Lemma 3.3.6. (1) is clear since $L_2(r_0, i(r))$ is a subset of $L_2(\mathcal{A}_l)$ and $L_2(r_0, j(r))$ is a superset of $L_2(\mathcal{A}) \setminus L_2(\mathcal{A}_l)$. Proposition 7.2.1 implies $i(r)$ is the minimum element of the fiber $f^{-1}(r)$. Dually, $j(r)$ is the maximum element of $f^{-1}(r)$. This verifies (2). Finally, (3) follows from the uniqueness of X_0. ∎

7.3 Main Theorem

Theorem 7.3.1 *Let $\mathcal{A}$ be a supersolvable arrangement with chamber c_0 and gallery r_0 incident to a modular flag. Let $\underline{x} = (x_1, x_2, \ldots, x_m) \in \omega(\mathcal{A}, c_0)$ be a cellular string of $\mathcal{A}$. The set of galleries incident to x forms a closed interval of $\operatorname{Gal}(\mathcal{A}, r_0)$ whose proper part is homotopy equivalent to a sphere of dimension $\sum_{i=1}^m \operatorname{codim}(x_i) - m - 2$.*

To decompose the set of galleries incident to a cellular string into a product of bounded posets, we make use of the following simple lemma.

Lemma 7.3.2 *Assume $\mathcal{A}$ is supersolvable with modular flag of faces $\mathcal{F}:\ F_0 < F_1 < \cdots < F_d = 0$. If $X \in L(\mathcal{A})$ then the localized arrangement $\mathcal{A}_X$ is supersolvable with modular flag*

$$\mathcal{F}_X :\ (F_0)_X \leq (F_1)_X \leq \cdots \leq (F_d)_X = 0.$$

If r_0 is the unique gallery incident to $\mathcal{F}$, then $(r_0)_X$ is the unique gallery incident to $\mathcal{F}_X$.

Proof: Let $X \in L(\mathcal{A})$. If Y is modular in $L(\mathcal{A})$, then $X + Y \in L(\mathcal{A})$, so $X + Y \in L(\mathcal{A}_X)$. Moreover, if $Z \in L(\mathcal{A}_X)$, then $(X+Y)+Z = Y + (X+Z) = Y + Z \in L(\mathcal{A})$ so it is in $L(\mathcal{A}_X)$. Hence, $X + Y$ is a modular element of $L(\mathcal{A}_X)$. If $F \in \mathcal{L}(\mathcal{A})$ such that

F^0 is modular, then $F^0_X = F^0 + X$, which is modular in $L(\mathcal{A}_X)$. In particular, $\mathcal{F}_X$ is a flag of modular faces.

We next show that $\mathcal{F}_X$ is maximal. Suppose $(F_{i+1})_X$ is a proper face of $(F_i)_X$. Then

$$\begin{aligned}
1 &\leq \operatorname{codim}(X + F^0_{i+1}) - \operatorname{codim}(X + F^0_i) \\
&= (\operatorname{codim}(X) + \operatorname{codim}(F^0_{i+1}) - \operatorname{codim}(X \cap F^0_{i+1})) - (\operatorname{codim}(X) + \operatorname{codim}(F^0_i) - \operatorname{codim}(X \cap F^0_i)) \\
&= 1 + (\operatorname{codim}(X \cap F^0_i) - \operatorname{codim}(X \cap F^0_{i+1})) \\
&\leq 1.
\end{aligned}$$

Hence, the flag $\mathcal{F}_X$ is maximal.

Let r_0 be the unique gallery incident to $\mathcal{F}$ from F_0 to $-F_0$. Then $(r_0)_X$ crosses the hyperplanes in $(F_0)_X$ first, followed by the rest of $(F_1)_X$, followed by $(F_2)_X$, etc. As $\mathcal{A}_X$ is supersolvable with modular flag $\mathcal{F}_X$, this implies $(r_0)_X$ incident to $\mathcal{F}_X$. Uniqueness is immediate from Proposition 4.3.1. ∎

Proof: (of theorem) For each i, let $X_i = x^0_i$ and let $\mathrm{Gal}_i := \mathrm{Gal}(\mathcal{A}_{x_i}, (r_0)_{x_i})$ be a poset of galleries of a supersolvable arrangement with base gallery incident to a modular flag. By Theorem 7.2.2, the proper part of Gal_i is homotopy equivalent to $\mathbb{S}^{\operatorname{codim} x_i - 3}$.

A gallery r_i in Gal_i can be viewed as a gallery between $x_i \circ c_0$ and $x_i \circ (-c_0)$ in the arrangement $\mathcal{A}$. Since x is a cellular string, any sequence $(r_i \in \mathrm{Gal}_i)_i$ can be patched together to a gallery between c_0 and $-c_0$ in $\mathcal{A}$. Consequently, the galleries incident to $\underline{x}$ may be indentified with the product poset

$$\mathrm{Gal}(\mathcal{A}_{X_1}, (r_0)_{X_1}) \times \cdots \times \mathrm{Gal}(\mathcal{A}_{X_m}, (r_0)_{X_m}).$$

In general, if bounded posets P, Q are homotopy equivalent to spheres $\mathbb{S}^p, \mathbb{S}^q$, respectively, then $\overline{P \times Q}$ is homotopy equivalent to $\mathbb{S}^{p+q+2}$. Applying this fact to the above product completes the proof. ∎

Example 7.3.3 *Let $\mathcal{A}$ be a type A reflection arrangement with fundamental chamber c_0. The cellular strings $\omega(\mathcal{A}, c_0)$ may be identified with wiring diagrams where multiple wires may cross at a vertical section. The set of galleries incident to a given cellular string $\underline{x}$*

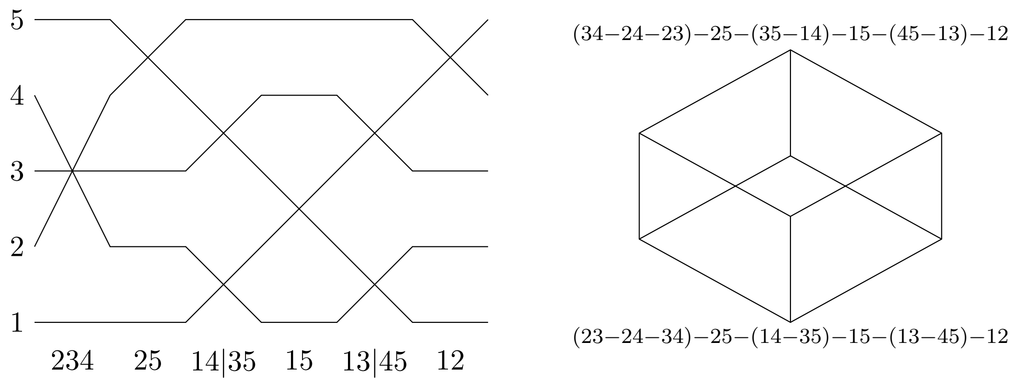


Figure 7.3: (Left) A cellular string of the type A_4 reflection arrangement. (Right) The interval of the gallery poset corresponding to this string. Its proper part is homeomorphic to $\mathbb{S}^1$.

are the simple wiring diagrams obtained by resolving multiple crossings in the diagram associated to $\underline{x}$*. The incident galleries form an interval in* $\mathrm{Gal}(\mathcal{A}, r_0)$ *whose minimum element resolves multiple crossings in colexicographic order and whose maximum element resolves in reverse colexicographic order. Figure 7.3 shows an example.*

Chapter 8

Lattice structure of Grid-Tamari orders

In Section 8.2, we define a Tamari-like poset, which we call a *Grid-Tamari order*. Denoted $\mathrm{GT}(\lambda)$, this is a partial order associated to a finite induced subgraph λ of the $\mathbb{Z} \times \mathbb{Z}$ grid. Grid-Tamari orders are defined as an acyclic orientation of the dual graph of a pure thin simplicial complex called a *non-kissing complex*. Special cases of Grid-Tamari orders include Tamari orders (λ is $2 \times n$ rectangle), Grassmann-Tamari orders (λ is a $k \times n$ rectangle) defined in Section 8.1, and type A Cambrian lattices (λ is a double-ribbon shape).

Our main result is

Theorem 8.0.4 *For any shape* λ*,* $\mathrm{GT}(\lambda)$ *is a congruence-uniform lattice.*

While this result was already known for Cambrian lattices, it was not even known that the Grassmann-Tamari orders were lattices. This was previously conjectured by Santos, Stump, and Welker [89, Conjecture 2.20].

We prove Theorem 8.0.4 by identifying $\mathrm{GT}(\lambda)$ as a lattice quotient of a poset of biclosed sets satisfying the conditions of Theorem 3.1.9. When λ is a $2 \times n$ rectangle, this reduces to the standard map from permutations to triangulations reviewed in §2.4.5. Our lattice quotient description allows us to compute the poset of lattice congruences of $\mathrm{GT}(\lambda)$, which we present in Theorem 8.7.1.

This chapter is organized as follows. In §8.1, we define the non-crossing complex and describe its geometric significance following [89]. In §8.2, we establish the purity and thinness of the non-kissing complex combinatorially, similar to the methodology employed in [89, Section 2.2] for proving purity and thinness of the non-crossing complex. We close the subsection by defining the orientation on the dual graph of the non-kissing complex whose transitive closure is a Grid-Tamari order. We emphasize that this directed graph is acyclic as a *consequence* of Theorem 8.0.4. A geometric proof of acyclicity in the non-crossing case appears in [89].

The reduced non-kissing complex is the boundary complex of a simplicial polytope. In §8.3, we describe a way to construct these polytopes by a sequence of edge-stellations and suspensions. It follows that the γ-vector of the non-kissing complex is the f-vector of a flag simplicial complex by results of [2] and [20]. Moreover, the Hasse diagram for the Grid-Tamari order is the 1-skeleton of the polar dual polytope. These dual polytopes may be constructed by dual operations, namely ridge-truncations and doublings. We remark that although ridge-truncations sometimes correspond to interval doublings, these two constructions do not match up in general.

In §8.4, we introduce a poset of biclosed subsets of segments in a shape λ. A collection of segments between two interior vertices of λ is *closed* if a segment s is in λ whenever there exists a partition of s into two subpaths that both lie in λ. We show that this closure satisfies the hypotheses given in Theorem 3.1.9, so its poset of biclosed sets is a congruence-uniform lattice.

A special lattice congruence on the lattice of biclosed sets of segments is presented in §8.5. In §8.6, we define a map η from biclosed sets of segments to the facets of the non-kissing complex, and show that the fibers of η are precisely the equivalence classes of this lattice congruence. We then deduce Theorem 8.0.4 by comparing the order induced by η with the Grid-Tamari order. Having established the congruence-uniformity of $\mathrm{GT}(\lambda)$, we compute the whole poset of lattice congruences in §8.7.

Some other interesting lattice quotients of the weak order called *Cambrian lattices* were introduced by Reading in [76]. In §8.8, we prove that the type A Cambrian lattices are examples of Grid-Tamari orders for double ribbon shapes. We prove this isomorphism using Reading's description of type A Cambrian lattices as a poset of triangulations of a polygon.

8.1 Non-crossing complexes

Fix $k, n \in \mathbb{N}$. Two sets $I, J \in \binom{[n]}{k}$ are *crossing* if $i_t < j_t < i_{t+1} < j_{t+1}$ for some t where $I - J = \{i_1 < \cdots < i_l\}$ and $J - I = \{j_1 < \cdots < j_l\}$. The sets I, J are *non-crossing* otherwise. For example, $\{1,4,5\}$ and $\{2,3,6\}$ are non-crossing, whereas $\{1,4,5\}$ and $\{2,4,6\}$ are crossing. The *non-crossing complex* $\Delta_{k,n}^{NC}$ is the collection of all pairwise non-crossing subsets of $\binom{[n]}{k}$.

For $l \geq 1$, let C_l be a chain poset with l elements. The complex $\Delta_{k,n}^{NC}$ may be realized as a regular, unimodular, Gorenstein triangulation of the order polytope $\mathcal{O}_{k,n}$ on $C_k \times C_{n-k}$; i.e., the polytope in $\mathbb{R}^{k(n-k)}$ defined by the inequalities $0 \leq x_{i,j} \leq 1,\ x_{i,j} \leq x_{i+1,j}$, and $x_{i,j} \leq x_{i,j+1}$ for $1 \leq i \leq k,\ 1 \leq j \leq n-k$ ([69, Theorem 8.1] or [89, Theorem 1.7]). This triangulation of $\mathcal{O}_{k,n}$ is distinct from the equatorial triangulation defined in [84], which is not flag in general. As a consequence of this geometric realization, after removing cone points, $\Delta_{k,n}^{NC}$ is a pure, thin complex of dimension $(k-1)(n-k-1)-1$. Moreover, there exists a simple polytope, the *Grassmann-associahedron*, with facial structure anti-isomorphic to $\Delta_{k,n}^{NC}$. As a flag, simplicial polytope, one may expect that the dual Grassmann-associahedron may be constructed by a sequence of suspensions and edge-stellations, which we prove in Section 8.3.

Any triangulation of $\mathcal{O}_{k,n}$ naturally gives rise to a monomial basis for the coordinate ring of the Grassmannian, the $\mathbb{C}$-algebra generated by the maximal minors of a $k \times n$ matrix of indeterminates (x_{ij}) [69]. Namely, a monomial $\prod_1^r x_{I_j}$ is in the basis if $\{I_1, \ldots, I_r\}$ is a face of the triangulation. The classical standard basis for this algebra is indexed by semistandard Young tableaux. The columns of a semistandard Young tableaux satisfy a compatibility condition that resembles a non-nesting analogue of the non-crossing condition defined above. Thus these two bases may be viewed as "opposite" in some sense; see [89, Remark 4.7]. One may hope to develop a straightening law for these monomials, though we do not pursue this here.

Let $\mathcal{J}$ be the set of order ideals of $C_k \times C_{n-k}$. The *Hibi ideal* is the ideal generated by $\{x_I x_J - x_{I\cap J} x_{I \cup J} :\ I, J \in \mathcal{J})$ in the polynomial ring on $\{x_I :\ I \in \mathcal{J}\}$. By results of [93], regular unimodular triangulations of $\mathcal{O}_{k,n}$ are in bijection with squarefree monomial initial ideals of the Hibi ideal. As observed in the introduction of [89], the triangulation induced by $\Delta_{k,n}^{NC}$ corresponds to a particularly nice initial ideal. We refer to the survey

[23, Section 6] for more background on Hibi ideals.

There is a natural orientation on the dual graph of $\Delta_{k,n}^{NC}$. If two facets $F_1 = F \cup \{I\}, F_2 = F \cup \{J\}$ are adjacent, then there is a unique index t for which $i_t < j_t < i_{t+1} < j_{t+1}$ where $I - J = \{i_1 < \cdots < i_l\}$ and $J - I = \{j_1 < \cdots < j_l\}$. We orient the edge $F_1 \to F_2$ if the pair $\{i_t, i_{t+1}\}$ is lexicographically smaller than $\{j_t, j_{t+1}\}$. For example, $\{145, 146, 236, 245\}$ and $\{146, 236, 245, 246\}$ are adjacent facets of $\Delta_{3,6}^{NC}$ with orientation $\{145, 146, 236, 245\} \to \{146, 236, 245, 246\}$ since 145 and 246 cross at 15 and 26. Defined by Santos, Stump, and Welker in [89], the *Grassmann-Tamari order* $\mathrm{GT}_{k,n}$ is the transitive closure of this relation. The smallest Grassmann-Tamari order not isomorphic to a Tamari lattice is drawn in Figure 1.4.

The non-crossing condition translates to a *non-kissing* condition on paths via the standard bijection between k-subsets of $[n]$ and paths in a $k \times (n-k)$ rectangle with South and East steps. For example, the set $\{1, 4, 5\}$ corresponds to the path from the NW-corner to the SE-corner of the rectangle such that the first, fourth, and fifth steps are to the South, while the others are to the East. Two paths p_1, p_2 in the plane are *kissing* if they agree on some subpath between vertices v and v' such that

1. p_1 enters v from the West and leaves v' to the South, and

2. p_2 enters v from the North and leaves v' to the East.

An example of two kissing paths is given in Figure 8.1. The non-kissing complex $\Delta^{NK}(\lambda)$ associated to a (possibly not rectangular) shape λ is the collection of pairwise non-kissing paths supported by λ. A poset $\mathrm{GT}(\lambda)$ analogous to the Grassmann-Tamari orders may be defined on the facets of this complex. We call $\mathrm{GT}(\lambda)$ the *Grid-Tamari order*; see Section 8.2.

8.2 Non-kissing complexes

Let λ be a finite induced subgraph of the $\mathbb{Z} \times \mathbb{Z}$ square grid. We refer to such a graph as a *shape*. A vertex v is *interior* if λ contains the 2×2 grid centered at v. Any vertex of λ that is not interior is called a *boundary vertex*. We say v is a *SE-corner* if the vertices one step South or East of v are not in λ. If v is a vertex of λ, then $\lambda \setminus v$ is the subgraph of λ with v removed.

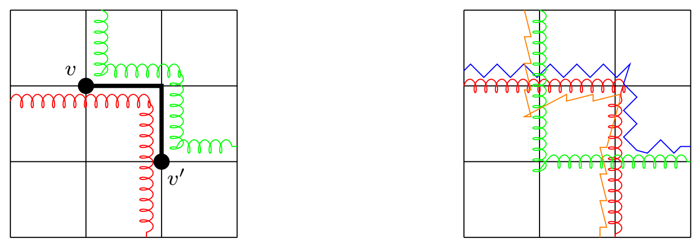


Figure 8.1: (left) Two paths kissing along the indicated segment from v to v'. The paths correspond to the sets 145 and 246, which are crossing. (right) A maximal family of non-kissing paths excluding horizontal and vertical paths.

A *path* supported by λ is a sequence of vertices $v_0, \ldots, v_t$ such that

- v_0 and v_t are boundary vertices,
- $v_1, \ldots, v_{t-1}$ are interior vertices, and
- v_i is one step South or East of v_{i-1} for all i.

Example 8.2.1 *For the path in Figure 8.2, the vertices w_0 and w_4 are boundary vertices, while w_1, w_2, and w_3 are interior. If v is removed from λ, then w_3 becomes a boundary vertex. The restriction of this path to $\lambda \setminus v$ is the sequence w_0, w_1, w_2, w_3.*

A path supported by λ is called a *segment* if its endpoints are also interior vertices. If s is a segment containing vertices v and v', then $s[v, v']$ denotes the sub-segment of s whose endpoints are v and v'. The initial (terminal) vertex of a segment s is denoted s_{init} (s_{term}). We abbreviate $s[s_{\text{init}}, v]$ and $s[v, s_{\text{term}}]$ to $s[\cdot, v]$ and $s[v, \cdot]$, respectively. A segment that only contains one vertex is called *lazy*. All other segments are *non-lazy*.

Two paths p_1, p_2 are *kissing* if they share vertices v, v' such that

- $p_1[v, v'] = p_2[v, v']$,
- p_1 enters v from the West and leaves v' to the South, and
- p_2 enters v from the North and leaves v' to the East.

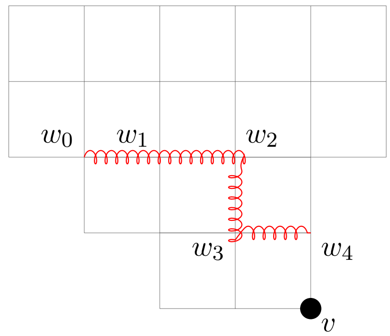


Figure 8.2: A shape with a path w_0, w_1, w_2, w_3, w_4 and a SE-corner v.

Otherwise p_1 and p_2 are *non-kissing*. The *non-kissing complex* $\Delta^{NK}(\lambda)$ is the (flag) simplicial complex whose faces are collections of pairwise non-kissing paths supported by λ. Let $\mathcal{F}(\Delta^{NK}(\lambda))$ denote the set of *facets*, the maximal faces of this complex. As horizontal and vertical paths are non-kissing with any path, we define the *reduced non-kissing complex* $\widetilde{\Delta}^{NK}(\lambda)$ to be the deletion of all horizontal and vertical paths from $\Delta^{NK}(\lambda)$.

Although a pair of non-kissing paths may twist around each other several times, there is a natural way to totally order paths that contain a specific edge. Let e be an edge of λ. If p_1 and p_2 are distinct non-kissing paths containing e, then they agree on some maximal segment $p_1[v, v']$ containing e. Order $p_1 \prec_e p_2$ if either p_1 enters v from the North or p_1 leaves v' to the South. A path $p \in F$ is the *bottom path* (*top path*) at an edge e if p is minimal (maximal) in F with respect to $\prec_e$.

Theorem 8.2.2 *Let F be a facet of $\Delta^{NK}(\lambda)$.*

1. *The map $e \mapsto \max_{\prec_e} F$ is a bijection between vertical edges of λ and non-horizontal paths in F.*

2. *Dually, the map $e \mapsto \min_{\prec_e} F$ is a bijection between horizontal edges of λ and non-vertical paths in F.*

3. *For paths $p \in F$ with at least one turn, there exists a unique path q distinct from p such that $F - \{p\} \cup \{q\}$ is non-kissing. Moreover, p and q kiss at a unique segment.*

Proof: For each of these statements, we proceed by induction on the size of λ. Let c be SE-corner of λ and let w be the point in $\mathbb{Z} \times \mathbb{Z}$ one step NW of c. If w is not an interior vertex of λ, then every path in λ is supported by $S \setminus c$, so the theorem holds by the inductive hypothesis. Hence, we may assume that w is an interior vertex of λ.

(1): We start by proving injectivity of the map. Suppose there is a path $p \in F$ that is on top at two distinct vertical edges e_1, e_2. Let v be the southern vertex of e_1 and let e be the edge west of v. Let $p' \in F$ be the bottom path at e. Define a path q supported by λ where $q[\cdot, v] = p'[\cdot, v]$ and $q[v, \cdot] = p[v, \cdot]$. Since $p \prec_{e_2} q$, the path q is not in F. Let t be the segment containing v along which p and p' agree. Since p and p' are non-kissing, p leaves t to the South and p' leaves to the East.

We claim that q is non-kissing with every path in F, contradicting the maximality of F. Indeed, if q and q' are kissing for some $q' \in F$, then they must kiss at a segment s containing v as q' is non-kissing with both p and p'.

If v is the initial vertex of s, then q' must leave the terminal vertex of s to the east while q leaves to the south. But this means q' enters v from the North, which contradicts maximality of p at e_1.

If v is not the initial vertex of s, then q' contains e. By the minimality of p' at e, q' must enter s from the West and leave s to the South. If t is a subsegment of s, then $q' \prec_e p'$, a contradiction. If t contains s, then q' and p are kissing, a contradiction.

Next we verify surjectivity. The restriction of paths in F to $S \setminus c$ defines a collection of non-kissing paths supported by $S \setminus c$. Let e_1 be the edge south of w and e_2 the edge east of w. It is straight-forward to check that if p and p' are distinct paths on λ with the same restriction to $S \setminus c$, then p must the top path at e_1 and p' the bottom path at e_2 (or vice versa). Hence, the map applied to $F \setminus c$ is still injective. By the inductive hypothesis, it is also surjective.

Now let q be a path in F not on top at e_1. Then the restriction of q to $S \setminus c$ is on top at some edge e. By the above computation, q is still on top at e. Hence, the map $e \mapsto \max_e F$ is surjective.

(2): This statement follows from part (1) by a dual argument.

(3): There exists a path r in $F - \{p\}$ on top at two vertical edges, say e_1 and e_2. Let v_1 be the South vertex of e_1 and v_2 be the North vertex of e_2. Let e_1' be the horizontal edge West of v_1 and e_2' the horizontal edge East of v_2.

Let r' be the bottom path at e_1' in $F-\{p\}$. We claim that $r'[v_1,v_2]=r[v_1,v_2]$ and that r' is the bottom path at e_2'.

Let v be the last vertex for which $r[v_1,v]=r'[v_1,v]$. Then $v\leq v_2$ since r is the top path at e_2. Let e be the vertical edge South of v, and let r_e be the top path at e. Choose v' minimal such that $r_e[v',v]=r[v',v]$. Since r_e and r' are non-kissing $v'\leq v_1$. However, as r is the top path at e_1, we either have $v_1=v'$ or $r_e=r$. If $r_e\neq r$, then $r_e\prec_{e_1'} r'$, a contradiction. Hence $r_e=r$ and $e=e_2$.

Let $r_{e_2'}$ be the bottom path at e_2'. Let v be the smallest vertex such that $r_{e_2'}[v,v_2]=r[v,v_2]$. Since r is the top path at e_1, $v_1\leq v$ holds. If $v_1<v$, then $r_{e_2'}$ enters v from the North while r' enters from the West. However this would force r and $r_{e_2'}$ to be kissing, a contradiction. Hence $v=v_1$ and $r'=r_{e_2'}$. This completes the proof of the claim.

Define paths q_{e_1},q_{e_2} such that $q_{e_1}[\cdot,v_2]=r[\cdot,v_2]$, $q_{e_1}[v_2,\cdot]=r'[v_2,\cdot]$ and $q_{e_2}[\cdot,v_2]=r'[\cdot,v_2]$, $q_{e_2}[v_2,\cdot]=r[v_2,\cdot]$. It is easy to check that $F-\{p\}\cup\{q_{e_i}\}$ is non-kissing for $i=1,2$. Moreover, q_{e_1} and q_{e_2} kiss along the unique segment $r[v_1,v_2]$. It remains to prove that these are the only two paths that are non-kissing with $F-\{p\}$.

Let q be a path such that $F-\{p\}\cup\{q\}$ is non-kissing. Then either q is on top at e_1 or e_2.

Assume q is on top at e_1. Let v be the largest vertex for which $q[v_1,v]=r[v_1,v]$. Since r is on top at e_2, $v\leq v_2$ holds. As q and r' are non-kissing, we must have $v=v_2$. If $q\neq q_{e_1}$, then they must kiss along some segment s. Since q is non-kissing with both r and r', this segment s must contain $r[v_1,v_2]$. Since $r\prec_{e_1} q$, q must enter s from the West and exit South.

Let v,v' be vertices such that $s=q[v,v']$. Let e be the edge North of v, and let p_e be the top path at e in $F-\{p\}\cup\{q_{e_1}\}$. Since p_e and q are non-kissing, we must have $p_e[v,v_1]=q[v,v_1]$. Hence, $p_e=q_{e_1}$, a contradiction. ∎

A simplicial complex is *pure* if its facets all have the same dimension. A pure complex is *thin* if every face of codimension 1 is contained in exactly two facets. From Theorem 8.2.2, we deduce the following corollary.

Corollary 8.2.3 *For any shape λ, the reduced non-kissing complex $\widetilde{\Delta}^{NK}(\lambda)$ is a pure, thin, flag simplicial complex.*

This result was proven in [69] and [89] for some specific shapes by identifying $\Delta^{NK}(\lambda)$ as a regular, unimodular triangulation of an order polytope with "enough" cone points. The regularity of this triangulation implies that $\widetilde{\Delta}^{NK}(\lambda)$ is the boundary complex of a polytope. When λ is a rectangle shape, this polytope is called the *Grassmann Associahedron* since the triangulation reflects many of the algebraic properties of the coordinate ring of the Grassmannian, and it reduces to the usual associahedron if λ has two rows [89]. In the next section, we give another proof of Corollary 8.2.3 and of polytopality for any shape λ by constructing $\widetilde{\Delta}^{NK}(\lambda)$ from the empty complex by a sequence of suspensions and edge-stellations.

Example 8.2.4 *We illustrate Theorem 8.2.2 with the facet* $F = \{145, 146, 236, 245\}$ *of* $\widetilde{\Delta}^{NC}_{3,6}$. *The sets in* F *correspond to the four non-kissing paths drawn in Figure 8.1. Including the two vertical paths* 234 *and* 345*, each of the six paths in* $F \cup \{234, 345\}$ *is the top path at a unique interior vertical edge.*

The unique facet distinct from F *containing* $F - \{145\}$ *is* $(F - \{145\}) \cup \{246\}$. *If one removes* 145 *from* F*, then* 245 *is on top at two different vertical edges. The segment supported by* 245 *between these two vertical edges is the unique segment along which the paths* 145 *and* 246 *kiss.*

The *dual graph* of a pure thin complex is the set of facets where two facets are adjacent if they intersect at a codimension 1 face. We define an orientation on the dual graph of $\widetilde{\Delta}^{NK}(\lambda)$ as follows. Let F_1, F_2 be adjacent facets, and let $p_1 \in F_1 - F_2,\ p_2 \in F_2 - F_1$. Then p_1 and p_2 are kissing at a unique segment, say $p_1[v, v']$. Orient the edge $F_1 \to F_2$ if p_1 enters v from the West (equivalently, p_1 leaves v' to the South). Let $\mathrm{GT}(\lambda)$ be the transitive closure of this relation.

Theorem 8.2.5 (see [89], Theorem 2.17) $\mathrm{GT}(\lambda)$ *is a partially ordered set.*

We call $\mathrm{GT}(\lambda)$ the *Grid-Tamari order*. When λ is a $2 \times n$ rectangle, $\mathrm{GT}(\lambda)$ is the usual Tamari lattice. For general λ, Theorem 8.2.5 is far from obvious. In [89], it is proved for all rectangle shapes by identifying $\mathrm{GT}(\lambda)$ with a poset of facets of a regular triangulation of a polytope, whose order is induced by a generic linear functional. We establish Theorem 8.2.5 as a consequence of Theorem 8.0.4.

8.3 Polytopal realization of the non-kissing complex

There are several known constructions of the dual associahedron by iteratively stellating faces of a simplex or cross-polytope [61], [20], [42]. We produce a similar construction for the non-kissing complex.

Fix a shape λ, and let c be a SE-corner of λ. Let v be the point one step NW of c. If v is not an interior vertex of λ, then $\widetilde{\Delta}^{NK}(\lambda \setminus c) = \widetilde{\Delta}^{NK}(\lambda)$. On the other hand, if v is interior, we construct a sequence of complexes $\Gamma_0, \ldots, \Gamma_l$ such that

- Γ_0 is isomorphic to the suspension of $\widetilde{\Delta}^{NK}(\lambda \setminus c)$,
- $\Gamma_l = \widetilde{\Delta}^{NK}(\lambda)$, and
- Γ_i is the stellation of Γ_{i-1} at some edge for all i.

Given a path in $\lambda \setminus c$, we extend it (uniquely) to a path in λ that does not turn at v. Then two paths in $\lambda \setminus c$ are non-kissing if and only if their extensions to λ are non-kissing. Let Γ_0 be the suspension of $\widetilde{\Delta}^{NK}(\lambda \setminus c)$ where the two new vertices correspond to the two paths q_W, q_N that only turn at v, where q_W enters v from the West and q_N enters v from the North.

Let e_W be the horizontal edge West of v, and let e_N be the vertical edge North of v. Let $p_1, \ldots, p_k$ be the list of paths distinct from q_W that turn at v and contain e_W, ordered so that if $i < j \leq k$ then $p_i \prec_{e_W} p_j$. This is well-defined since $<_{e_W}$ is a total order on these paths. Similarly, let $p_{k+1}, \ldots, p_l$ be the list of paths that turn at v and contain e_N, ordered so that if $k < i < j$ then $p_i \succ_{e_N} p_j$. For each i, let r_i be the same path as p_i except that it continues straight through v.

Then for each $i \leq k$, define Γ_i recursively as the complex $\mathrm{st}_{\{r_i, q_W\}}(\Gamma_{i-1})$, where the new vertex is labeled p_i. For $i > k$, we define Γ_i as the complex $\mathrm{st}_{\{r_i, q_N\}}(\Gamma_{i-1})$, where the new vertex is again labeled p_i.

With the above set-up, the following result is elementary, if somewhat tedious to verify.

Theorem 8.3.1 $\Gamma_l = \widetilde{\Delta}^{NK}(\lambda)$.

Proof: Let p, q be two paths supported by λ. We prove that p and q are adjacent in Γ_l if and only if they are non-kissing.

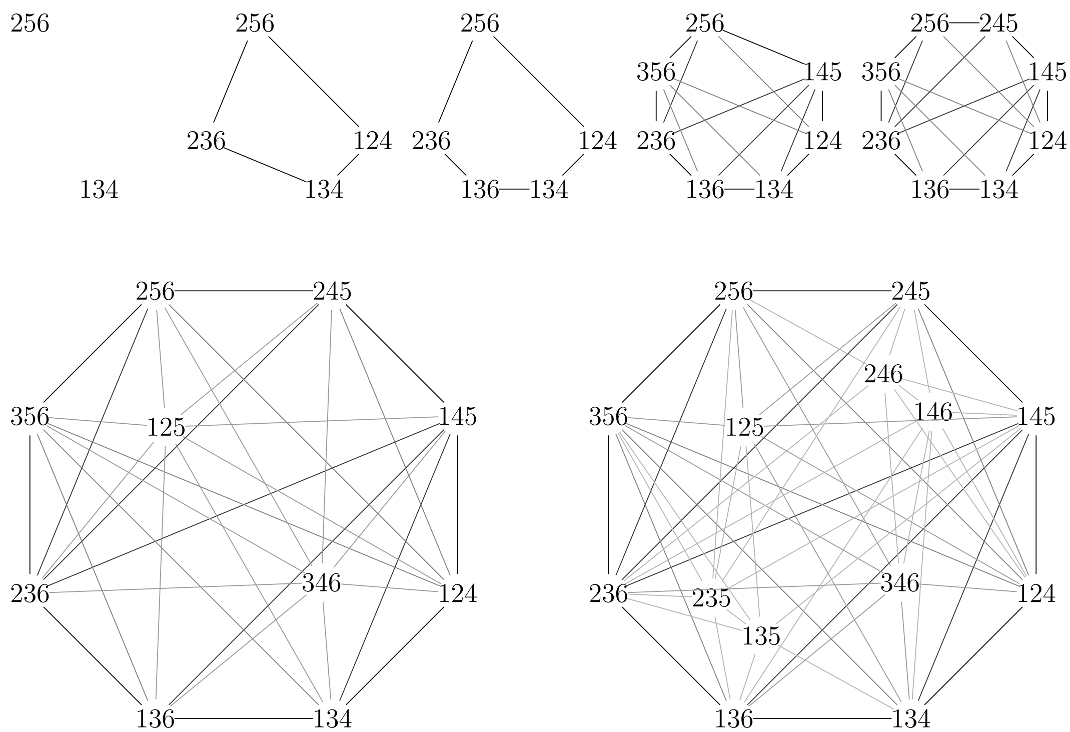


Figure 8.3: A construction of the reduced non-crossing complex $\widetilde{\Delta}^{NC}_{3,6}$ by a sequence of suspensions and edge stellations.

If neither p nor q turns at v, then p and q are adjacent in Γ_0 if and only if they are non-kissing. As these edges are not stellated by the construction, it follows that p and q are adjacent in Γ_l exactly when they are non-kissing.

Assume $q = q_W$. Then q kisses p only if p leaves v to the East. If p and q kiss at (v), then either $p = q_N$ or $p = q_i$ for some $i > k$. In either case, they are not adjacent in Γ_l. If p and q kiss at a segment s containing e_W, then $p = r_i$ for some $i \leq k$. In this case, p and q are separated in Γ_i. As q is adjacent to every other vertex of Γ_l, we are done in this case. A similar argument holds if $q = q_N$.

Now assume $p = p_i$ for some $i \leq k$. Suppose p and q kiss along a segment s not containing v. Then r_i and q also kiss along s. If q does not turn at v, then r_i and q are not adjacent in Γ_0, so p and q are not adjacent in Γ_i. If q does turn at v, then $q = p_j$. Without loss of generality, we may assume $i < j$. Then r_i and r_j are not adjacent, so p and r_j are not adjacent in Γ_i and p and q are not adjacent in Γ_j.

Assume $p = p_i$ for some $i \leq k$ and suppose p and q only kiss along a segment s containing v. If $s = (v)$, then $q = p_j$ for some $j > k$ or $q = q_N$. In either case, p and q are not adjacent in Γ_l. If s contains e_W then $q = r_j$ for some $j \leq k$. As $p_j \prec_{e_W} p_i$ we deduce that $j < i$. Hence, q_W is not adjacent to q in Γ_{i-1}, so p and q are not adjacent in Γ_i.

Now assume $p = p_i$ for some $i \leq k$ and suppose p and q are non-kissing. If q does not contain v, then q is adjacent to r_i and q_W in Γ_0, so p and q are adjacent in Γ_i. If q contains v, then either $q = q_W$, $q = r_j$ for some $j > k$, $q = r_j$ for some $j \leq k$, or $q = p_j$ for some $j \leq k$. The first case has already been handled. In the second case, r_j and r_i are non-kissing, so p and q are adjacent in Γ_i. In the third case, either r_i and r_j are non-kissing, or $i < j$; for both situations, p and q are adjacent in Γ_i. Finally, if $q = p_j$ for some $j \leq k$, we may assume $i < j$ without loss of generality. Then p and r_j are adjacent in Γ_i, so p and q are adjacent in Γ_j.

A similar argument holds when $p = p_i$ and $i > k$. This completes the proof. ∎

8.4 Biclosed Sets of Segments

Fix a shape λ and let S denote the set of segments supported by λ. Two segments s and t are *composable* if s_{term} is one unit North or West of t_{init}. If s and t are composable, then the composite $s \circ t$ is the segment containing both s and t. Given a set X of segments of λ, say X is *closed* if for $s, t \in S$, $s, t \in X$ and $s \circ t \in S$ implies $s \circ t \in X$; see Figure 8.4. We let $\text{Bic}(S)$ denote the poset of biclosed sets of segments.

This closure on segments may be realized as a 2-closure for a certain real vector configuration. A *cell* of λ is a unit square whose four corners are all vertices of λ. Let $\text{Cell}(\lambda)$ denote the set of cells of λ. To each interval vertex v of λ, we associate the vector $f_v \in \mathbb{R}^{\text{Cell}(\lambda)}$ where for a cell c,

$$f_v(c) = \begin{cases} 1 \text{ if } v \text{ is the SE or NW corner of } c \\ -1 \text{ if } v \text{ is the SW or NE corner of } c \\ 0 \text{ otherwise.} \end{cases}$$

For segments $(v_1, \ldots, v_l) \in S$, set $f_{(v_1,\ldots,v_l)} = \sum_i f_{v_i}$. It is easy to verify that segments s, t are composable if and only if there exists a segment u such that $f_u = f_s + f_t$.

Example 8.4.1 *Suppose λ is a $2 \times n$ rectangle. Labeling the interior vertices $1, \ldots, n-1$ from left to right, a segment s may be identified with the set $\{i, j\} \in \binom{[n]}{2}$ where i is the label on s_{init} and $j-1$ is the label on s_{term}. The closure on segments then agrees with the closure on $\binom{[n]}{2}$ defined in Example 3.1.8. Hence,* $\text{Bic}(S)$ *is isomorphic to the weak order on permutations of $[n]$. Moreover, the vector configuration $\{\frac{1}{\sqrt{2}} f_s : s \in S\}$ is the set of positive roots of a root system of type A_{n-1}.*

Remark 8.4.2 *The vectors f_s for $s \in S$ are called* bending vectors *in [89]. Their significance is explained in [89, Lemma 4.9]: If $F \xrightarrow{s} F'$ are adjacent facets of $\Delta^{NC}_{k,n}$, viewed as a triangulation of the order polytope on a product of chains, then f_s is orthogonal to the ridge $F \cap F'$ with F' on the positive side.*

Given this result, we are led to consider the bending arrangement $\mathcal{A}_\lambda = \{H_s : s \in S\}$ *where H_s is the hyperplane orthogonal to f_s. Following [75], we may expect that*

1. $\text{Ch}(\mathcal{A}_\lambda)$ *is a lattice,*

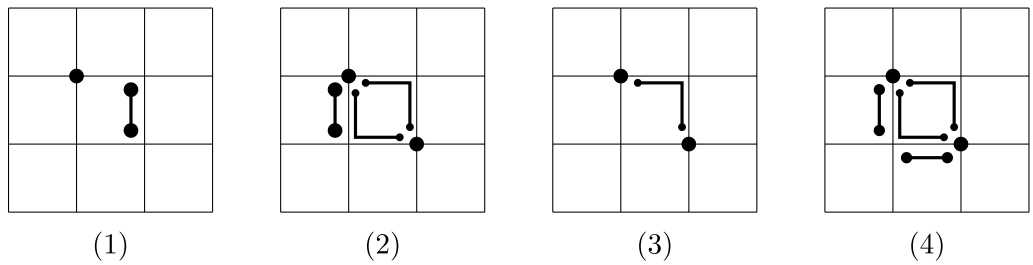


Figure 8.4: (1) Two composable segments. (2) A biclosed set X of five segments. (3) $X^{\downarrow}$. (4) $X^{\uparrow}$.

2. $\mathrm{GT}(\lambda)$ *is a lattice quotient of* $\mathrm{Ch}(\mathcal{A}_\lambda)$,

3. $\mathrm{GT}(\lambda)$ *is a fan poset on some complete fan* $\mathcal{F}$*, which is refined by the arrangement fan, and*

4. $\mathcal{F}$ *is the normal fan of a simple polytope.*

However, (1) is not *true when* λ *contains a* 3×3 *square. The chamber poset naturally injects into the poset of biclosed sets, which we prove is a lattice in Corollary 8.4.6. Replacing the chamber poset by the poset of biclosed sets, (2) is a restatement of Corollary 8.6.11. (3) seems to follow from results of [89], though we are not sure. We consider (4) to be an interesting open problem.*

Lemma 8.4.3 *If* $X \in \mathrm{Bic}(S)$ *and* c *is a SE-corner of* λ*, then* $X \setminus c$ *is a biclosed set of segments of* $S \setminus c$*.*

Proof: Let s, t, u be segments supported by λ such that $s \circ t = u$.

If $s, t \in X \setminus c$ then u is supported by $\lambda \setminus c$. Since X is closed, we conclude $u \in X \setminus c$.

If $u \in X \setminus c$ then both s and t are supported by $S \setminus c$. Since $S - X$ is closed, either s or t is in $X \setminus c$. ■

The following description of the closure is immediate from the definition. We record it here since it is a useful tool in later sections.

Lemma 8.4.4 *For* $X \subseteq S$*,* $\overline{X}$ *is the set of segments* s *such that there exist segments* $s_1, \ldots, s_l \in X$ *with* $s = s_1 \circ \cdots \circ s_l$*.*

We partially order S by inclusion; that is, $s \subseteq t$ means s is a subsegment of t.

Theorem 8.4.5 *If λ is any shape, then*

1. $\mathrm{Bic}(S)$ *is ordered by single-step inclusion,*

2. $W \cup \overline{(X \cup Y) - W}$ *is biclosed for* $W, X, Y \in \mathrm{Bic}(S)$ *with* $W \subseteq X \cap Y$*, and*

3. *if* $s, t, u \in S$ *such that* $s \circ t = u$*, then* $s \subsetneq u$ *and* $t \subsetneq u$*.*

Proof: We prove the theorem by induction on the size of λ. Let c be a SE-corner of λ. We may assume that the vertex w NW of c is an interior vertex as otherwise $S = S \setminus c$.

(1): Let $X, Y \in \mathrm{Bic}(S)$ such that $X \subsetneq Y$. If $s \in Y - X$ is of minimum length, then for any splitting $s = t \circ u$, either $t \in X$ or $u \in X$. Let $s \in Y - X$ be of maximum length such that for any splitting $s = t \circ u$, either $t \in X$ or $u \in X$. We prove that $X \cup \{s\}$ is biclosed.

If $X \cup \{s\}$ is not biclosed, then there exists $t \in X$ such that $s \circ t$ or $t \circ s$ is in $S - X$. Among such segments t, choose one of minimum length. Without loss of generality, we may assume $s \circ t$ is in $S - X$. Then $s \circ t \in Y$ since Y is closed. By maximality of s, there exists a splitting $s' \circ t' = s \circ t$ such that s' and t' are not in X. We distinguish two cases:

(a) Assume s' is an initial segment of s. Then $s = s' \circ u$ and $t' = u \circ t$ for some segment u. By assumption on s, we have $u \in X$. Since X is closed, this forces $t' \in X$, a contradiction.

(b) Assume s is an initial segment of s'. Then $s' = s \circ u$ and $t = u \circ t'$ for some segment u. Since $t \in X$, $t' \notin X$, we deduce $u \in X$. Since u is shorter than t, we deduce $s' \in X$, a contradiction.

Hence, $X \cup \{s\}$ is biclosed.

(2): Let $W \in \mathrm{Bic}(S)$. Assume, for $W' \in \mathrm{Bic}(S)$ with $W \subsetneq W'$:

$$W' \cup \overline{(X \cup Y) - W'} \text{ is biclosed for } X, Y \in \mathrm{Bic}(S) \text{ with } W' \subseteq X \cap Y.$$

Let $X, Y \in \mathrm{Bic}(S)$ such that $W \subseteq X \cap Y$. We may assume that W is a maximal biclosed set contained in $X \cap Y$. If $X \subseteq Y$, the result is immediate. If X and Y are incomparable, then by (1), there exists $s \in X - W,\ t \in Y - W$ such that $W \cup \{s\}$ and $W \cup \{t\}$ are biclosed.

Set $Z = W \cup \overline{\{s, t\}}$. If s and t are not composable, then $Z = W \cup \{s, t\}$ is biclosed. If $s \circ t = u$, we claim that $Z = W \cup \{s, t, u\}$ is biclosed.

If Z is not closed, then there exists $v \in W$ such that $v \circ u$ or $u \circ v$ is in $S - W$. We may assume without loss of generality that $v \circ u$ is in $S - W$. Since $W \cup \{s\}$ is closed and $v \circ s \in S$, we have $v \circ s \in W$. But $W \cup \{t\}$ is closed, so $v \circ s \circ t$ is in W, a contradiction. Hence, Z is closed.

If $S - Z$ is not closed, then there exists a splitting $s' \circ t' = u$ such that s' and t' are in $S - Z$. Then either s is an initial subsegment of s' or t is a terminal subsegment of t'. Without loss of generality, we may assume s is an initial subsegment of s'. Then there exists a segment u' with $s \circ u' = s'$ and $u' \circ t' = t$. Since $W \cup \{s\}$ is closed, the condition $s \circ u' = s'$ implies $u' \notin W$. However, as $S - (W \cup \{t\})$ is closed, the latter condition implies $u' \in W$, a contradiction.

Therefore, Z is biclosed. Applying the assumption with $W' = W \cup \{s\}$, we deduce that

$$W \cup \{s\} \cup \overline{(X \cup Z) - (W \cup \{s\})}$$

is biclosed. Similarly,

$$W \cup \{t\} \cup \overline{(Y \cup Z) - (W \cup \{t\})}$$

is biclosed. As both of these sets contain Z, we deduce that

$$Z \cup \overline{((W \cup \{s\} \cup \overline{(X \cup Z) - (W \cup \{s\})}) \cup (W \cup \{t\} \cup \overline{(Y \cup Z) - (W \cup \{t\})})) - Z}$$

is biclosed. This set is equal to

$$Z \cup \overline{\overline{(X \cup Y \cup Z) - W} - Z}.$$

But,

$$X \cup Y \subseteq Z \cup \overline{\overline{(X \cup Y \cup Z) - W} - Z} \subseteq W \cup \overline{(X \cup Y \cup Z) - W} = W \cup \overline{(X \cup Y) - W} \subseteq \overline{X \cup Y}.$$

Since $\overline{X \cup Y}$ is the smallest closed set containing $X \cup Y$, we deduce the equality

$$Z \cup \overline{\overline{(X \cup Y \cup Z) - W} - Z} = \overline{X \cup Y}.$$

Hence, $W \cup \overline{(X \cup Y) - W}$ is biclosed, as desired.

(3): This is immediate from the definitions. ∎

Applying Theorem 3.1.9, we deduce

Corollary 8.4.6 $\mathrm{Bic}(S)$ *is a congruence-uniform lattice.*

Remark 8.4.7 *The hypotheses of Theorems 3.1.7 and 3.1.9 were chosen with two examples in mind, namely the 2-closure on finite root systems and the closure operator defined in this section. For the 2-closure on a real simplicial hyperplane arrangement, the first two hypotheses hold, but the third may not. In this case, a weaker version of the acyclic condition is enough to prove congruence-normality [74, Theorem 25].*

8.5 A quotient of $\mathrm{Bic}(S)$

Given a biclosed set X of segments, let $X^{\downarrow}$ be the set of segments s in X such that t is in X whenever t is a SW-subsegment of s. Let $X^{\uparrow}$ be the set of segments s such that there exists t in X that is a NE-subsegment of s. An example is shown in Figure 8.4.

Transposition of shapes $\lambda \to \lambda^{\mathrm{tr}}$ induces a map on segments $s \mapsto s^{\mathrm{tr}}$. Given a set X of segments of λ, we let X^{tr} denote the set of transposed segments of λ^{tr}. Transposition commutes with complementation. Let $X^{c\,\mathrm{tr}}$ be the composition of these two involutions.

Claim 8.5.1 *For $X \subseteq S$,*

$$(X^{\uparrow})^{c\,\mathrm{tr}} = (X^{c\,\mathrm{tr}})^{\downarrow}.$$

Proof: A segment s is in $(X^{\uparrow})^{c\,\mathrm{tr}}$ if and only if s^{tr} is not a segment in $X^{\uparrow}$. But this holds exactly when none of the NE-subsegments of s^{tr} are in X. This occurs if none of the SW-subsegments of s are in X^{tr}, which is equivalent to $s \in (X^{c\,\mathrm{tr}})^{\downarrow}$. ∎

Claim 8.5.2 *If X is biclosed, then*

1. *X^{tr} is biclosed,*

2. *X^{c} is biclosed,*

3. *$X^{\downarrow}$ is biclosed, and*

4. *$X^{\uparrow}$ is biclosed.*

Proof: Parts (1) and (2) are immediate from the definitions. Part (4) follows from (1)-(3) with Claim 8.5.1. We verify part (3).

Let $s, t \in X^{\downarrow}$ such that $s \circ t$ is a segment. Since X is biclosed, $s \circ t$ is in X. If u is a SW-subsegment of $s \circ t$, then either $u \subseteq s$, $u \subseteq t$ or neither inequality holds. In the first two cases, it follows that $u \in X$ from $s, t \in X^{\downarrow}$. In the remaining case, we may divide u into two pieces $u = u_1 \circ u_2$ where u_1 is a SW-subsegment of s and u_2 is a SW-subsegment of t. Hence, $u_1, u_2 \in X$, so also $u \in X$. Therefore, $X^{\downarrow}$ is closed.

On the other hand, if $s \in X^{\downarrow}$ such that $s = t \circ u$, then either t or u is a SW-subsegment. Hence, $X^{\downarrow}$ is co-closed as well. ∎

Claim 8.5.3 *The maps $X \mapsto X^{\downarrow}$ and $X \mapsto X^{\uparrow}$ are idempotent and order-preserving.*

Proof: The order-preserving assertion is immediate from the definition. It remains to prove the maps are idempotent.

For segments s, t, u, if s is a SW-subsegment of t and t is a SW-subsegment of u, then s is a SW-subsegment of u. Hence, for $u \in X$, if every subsegment of u is in X, then every subsegment of u is also in $X^{\downarrow}$. The claim follows immediately. ∎

Claim 8.5.4 $(X^{\downarrow})^{\uparrow} = X^{\uparrow}$. *Dually,* $(X^{\uparrow})^{\downarrow} = X^{\downarrow}$.

Proof: The forward inclusion $(X^{\downarrow})^{\uparrow} \subseteq X^{\uparrow}$ follows from Claim 8.5.3.

If $s \in X^{\uparrow}$, then there exists $t_0 \in X$ such that t_0 is a NE-subsegment of s. If $t_0 \notin X^{\downarrow}$, then there exists a SW-subsegment u_0 of t_0 that is not in X. Then $t_0 = u_0' \circ u \circ u_0''$ where u_0' and u_0'' are (possibly empty) NE-subsegments of t_0. Since X is biclosed, either u_0' or u_0'' is in X. In particular, t_0 has a NE-subsegment t_1 that is in X. Continuing in this manner, we produce a segment $t \in X^{\downarrow}$ that is a NE-subsegment of s. Hence, $s \in (X^{\downarrow})^{\uparrow}$, as desired.

The second claim follows from the first via Claim 8.5.1. ∎

Let Θ be the equivalence relation on $\mathrm{Bic}(S)$ where $X \equiv Y \mod \Theta$ if $X^{\downarrow} = Y^{\downarrow}$. By Lemma 3.1.4, we deduce the following result.

Theorem 8.5.5 *Θ is a lattice congruence on* $\mathrm{Bic}(S)$.

Example 8.5.6 *Let λ be the $2 \times n$ rectangle from Example 8.4.1. If X is a biclosed subset of S, then $X^{\downarrow}$ is the set obtained by removing horizontal segments for which some initial part is not in X. The set $X^{\uparrow}$ is obtained by adding horizontal segments to X for which some initial part is not in X but the corresponding terminal part is in X. By this observation it follows that $X^{\uparrow}$ is the largest biclosed set for which $(X^{\uparrow})^{\downarrow} = X^{\downarrow}$. In particular, the equivalence classes are all closed intervals of the form $[X^{\downarrow}, X^{\uparrow}]$ for some $X \in \mathrm{Bic}(S)$. Moreover, $\pi^{\uparrow}(X) = X^{\uparrow}$ and $\pi_{\downarrow}(X) = X^{\downarrow}$, so $\pi^{\uparrow}$ and $\pi_{\downarrow}$ are both order-preserving maps, thus verifying Theorem 8.5.5 in this case. The argument for general shapes follows similar reasoning.*

When λ is a $2 \times n$ rectangle, the bijection in Example 8.4.1 takes biclosed sets X for which $X^{\downarrow} = X$ to inversion sets of 312-avoiding permutations. Indeed, if a permutation $\sigma = \sigma_1 \cdots \sigma_n$ contains a 312 pattern, say with values $i < j < k$, then the corresponding biclosed set X has a long segment labeled $\{i, k\}$ for which the initial part $\{i, j\}$ is not in X.

8.6 Proof of Theorem 8.0.4

For this section, we fix a shape λ, and let S denote the set of segments of λ. Furthermore, we let E_V denote the set of interior vertical edges in λ and let $\mathcal{P}$ be the set of paths supported by λ.

We define a function $\eta : \mathrm{Bic}(S) \to 2^{\mathcal{P}}$ as follows. Let $X \in \mathrm{Bic}(S)$ be given. If $e \in E_V$ is an edge from u to v, let p_e be the path such that for interior vertices $u' \in p_e[\cdot, u]$ and $v' \in p_e[v, \cdot]$:

(i) if $p_e[u', u]$ is (not) in X then p_e enters u' from the North (West); and

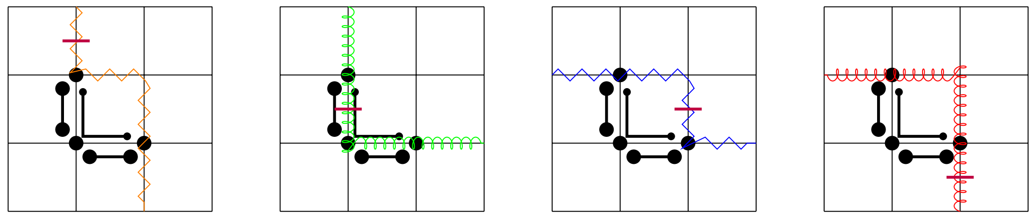


Figure 8.5: The four non-vertical and non-horizontal paths in $\eta(X)$ where X is the set of black segments.

(ii) if $p_e[v, v']$ is (not) in X then p_e leaves v' to the East (South).

Let $\eta(X)$ be the union of $\{p_e : \ e \in E_V\}$ with the set of horizontal paths supported by λ.

Example 8.6.1 *If X is the biclosed set of six black segments in Figure 8.5, each of the six interior vertical edges corresponds to a non-horizontal path in $\eta(X)$. In Figure 8.5, the four paths corresponding to the four marked purple edges are drawn. The other two vertical edges correspond to vertical paths. This is the same collection of paths as in Example 8.2.4.*

Claim 8.6.2 *Let p be a path in $\eta(X)$ containing a segment s.*

1. *If p enters s from the West and leaves s to the South, then s is not in X.*

2. *Similarly, if p enters s from the North and leaves s to the East, then s is in X.*

Proof: We prove 1. The proof of 2 is similar.

Let e be the interior vertical edge with $p = p_e$. Assume p enters s from the West and leaves s to the South. We prove that s is not in X by considering several cases.

If e is contained in s, then by construction $s[\cdot, e_{\text{init}}]$ and $s[e_{\text{term}}, \cdot]$ are not in X. If e precedes s in p, then $p[e_{\text{term}}, s_{\text{term}}]$ is not in X while $p[e_{\text{term}}, s_{\text{term}}] - s$ is in X. Finally, if e comes after s in p, then $p[s_{\text{init}}, e_{\text{init}}]$ is not in X while $p[s_{\text{init}}, e_{\text{init}}] - s$ is in X. In each case, we conclude that s is not in X since X is biclosed. ∎

Claim 8.6.3 *If e and e' are distinct interior vertical edges, then p_e and $p_{e'}$ are distinct paths.*

Proof: Assume to the contrary that p_e and $p_{e'}$ are the same. Without loss of generality, we may assume that e precedes e' in p_e. By definition of p_e, the segment $p_e[e_{\text{term}}, e'_{\text{init}}]$ is not in X. By the definition of $p_{e'}$, the same segment $p_{e'}[e_{\text{term}}, e'_{\text{init}}]$ is in X, a contradiction. ∎

Proposition 8.6.4 *$\eta(X)$ is a maximal collection of non-kissing paths.*

Proof: Suppose that $\eta(X)$ contains two paths p_{e_1}, p_{e_2} kissing along a common segment s. By Claim 8.6.2, s must be both in X and not in X, a contradiction. Hence, $\eta(X)$ is a set of non-kissing paths.

By Claim 8.6.3, $\eta(X)$ is of maximal size. ∎

By Proposition 8.6.4, η is a map from $\mathrm{Bic}(S)$ to $\mathcal{F}(\Delta^{NK}(\lambda))$.

Claim 8.6.5 *p_e is the top path at e in $\eta(X)$ (i.e. p_e is maximum with respect to the total order $\prec_e$ from Section 8.2).*

Proof: Let e' be an edge distinct from e such that $p_{e'}$ contains e. Without loss of generality, we may assume e precedes e' in $p_{e'}$. By definition of $p_{e'}$, X contains the segment $p_{e'}[e_{\text{term}}, e'_{\text{init}}]$.

Let s be the initial segment of $p_e[e_{\text{term}}, \cdot]$ along which p_e and $p_{e'}$ agree. Assume p_e leaves s to the South and $p_{e'}$ leaves to the East. By definition of p_e, this implies s is not in X. By definition of $p_{e'}$, the segment $p_{e'}[e_{\text{term}}, e'_{\text{init}}] - s$ is also not in X. As X is co-closed, this implies $p_{e'}[e_{\text{term}}, e'_{\text{init}}]$ is not in X, a contradiction. Therefore, p_e is the top path at e. ∎

Let F be a facet of $\Delta^{NK}(\lambda)$. For each path $p \in F$, let A_p be the set of SW-subsegments of p. That is, A_p consists of the segments $p[v, v']$ such that p enters v from the North and leaves v' to the East. Set $\phi(F) = \overline{\bigcup_{p \in F} A_p}$. A priori, ϕ is a map from facets of $\Delta^{NK}(S)$ to sets of segments.

Claim 8.6.6 *$\phi(F)$ is a biclosed set of segments.*

Proof: It is sufficient to show that A_p is a biclosed set as this would imply

$$\bigvee_{p\in F} A_p = \overline{\bigcup_{p\in F} A_p}.$$

No two segments in A_p are composable, so it is closed. Let $s \in A_p$, and let t, u be segments such that $t \circ u = s$. If the edge separating t and u is horizontal, then t is in A_p. If the edge separating t and u is vertical, then u is in A_p. Hence, A_p is biclosed. ■

Now we have defined functions $\eta : \mathrm{Bic}(S) \to \mathcal{F}(\Delta^{NK}(\lambda))$ and $\phi : \mathcal{F}(\Delta^{NK}(\lambda)) \to \mathrm{Bic}(S)$. We next show that η is surjective.

Claim 8.6.7 *The composite $\eta \circ \phi$ is equal to the identity on $\mathcal{F}(\Delta^{NK}(\lambda))$.*

Proof: Let $F \in \mathcal{F}(\Delta^{NK}(\lambda))$. Given a non-horizontal path p in F, we prove that p is in $\eta(\phi(F))$. Suppose p is on top at edge e. Let q be the path associated to e in $\eta(\phi(F))$. If $q = p$, we are done. Otherwise, we may assume that p and q are distinct after e. If not, then a similar argument may be used when p and q are distinct before e.

Let s be the longest segment along which p and q agree starting from e_{term}. If p leaves s to the East, then s is in A_p. This forces q to leave s to the East as well, contradicting the maximality of s. Hence, q leaves s to the East and p leaves s to the South.

Since s is in $\overline{\bigcup_{p\in F} A_p}$, the segment may be decomposed as $s = s_1 \circ \cdots \circ s_l$ where $s_i \in A_{p_i}$ for $i \in [l]$. If $s_1 \notin A_p$, then $p <_e p_1$, a contradiction. Let k be the smallest index for which $s_1 \circ \cdots \circ s_k \notin A_p$. Then p enters s_k from the West and leaves to the South. But $s_k \in A_{p_k}$, so p and p_k are kissing, a contradiction.

But this means $s \in A_p$, in contradiction to the assumption that p leaves s to the South. Hence, we conclude $p = q$. ■

Claim 8.6.8 *For $X \in \mathrm{Bic}(S)$, $\phi \circ \eta(X) = X^{\downarrow}$.*

Proof: We prove

(a) $X^{\downarrow} \subseteq \phi \circ \eta(X)$

(b) $\phi \circ \eta(X) \subseteq X$

(c) For $s \in \phi \circ \eta(X)$, if s' is a SW-subsegment of s, then $s' \in \phi \circ \eta(X)$.

The claim is then immediate.

(a) Let $s \in X^{\downarrow}$. We show s is in $\phi \circ \eta(X)$ by induction on the length of s. Let e be the vertical edge with terminal vertex s_{init}.

Let t be the initial subsegment of s that coincides with p_e in $\eta(X)$. If $s = t$, then p_e leaves s to the East, so s is in A_{p_e}. Assume $s = t \circ t'$ for some segment t'. If s leaves t to the East, then t is in X, so p_e also leaves to the East. Hence, s leaves t to the South while p_e leaves to the East. By the induction hypothesis, t' is in $\phi \circ \eta(X)$. Hence, s is in $\phi \circ \eta(X)$, as desired.

(b) Let $p \in \eta(X)$. It suffices to show that A_p is a subset of X. Suppose p is on top at some vertical edge e. By Claim 8.6.5, $p = p_e$ in the construction of $\eta(X)$. Fix $s \in A_p$. If s contains e, then since s is a SW-subsegment of p, X contains both $s[\cdot, e_{\text{init}}]$ and $s[e_{\text{term}}, \cdot]$. As X is closed, this implies $s \in X$. If e precedes s, then X contains $p[e_{\text{term}}, s_{\text{term}}]$ but not $p[e_{\text{term}}, s_{\text{term}}] - s$. As X is co-closed, this implies $s \in X$. If e appears after s, then X contains $p[s_{\text{init}}, e_{\text{init}}]$ but not $p[s_{\text{init}}, e_{\text{init}}] - s$. As X is co-closed, this forces $s \in X$. Therefore, we conclude that $A_p \subseteq X$ holds.

(c) Let $s \in \phi \circ \eta(X)$ and let s' be a SW-subsegment of s. We show that s' is in X. Then $s = s_1 \circ \cdots \circ s_l$ for some segments $s_i \in A_{p_i}$ and paths $p_i \in \eta(X)$. As s' is a subsegment of s, there exist indices $i \leq j$ for which $s' = s'_i \circ s_{i+1} \circ \cdots \circ s_{j-1} \circ s'_j$ where s'_i is a subsegment of s_i and s'_j is a subsegment of s_j. Since s' is a SW-subsegment of s, it follows that s'_i is a SW-subsegment of s_i and s'_j is a SW-subsegment of s_j. Hence, $s'_i \in A_{p_i}$ and $s'_j \in A_{p_j}$. We conclude that s' is in $\phi \circ \eta(X)$. ∎

Using Claim 8.6.8, it is easy to show that the fibers of η are equivalence classes of Θ.

Claim 8.6.9 *For $X, Y \in \mathrm{Bic}(S)$, $\eta(X) = \eta(Y)$ if and only if $X^{\downarrow} = Y^{\downarrow}$.*

Proof: Assume $\eta(X) = \eta(Y)$. Then

$$X^{\downarrow} = \phi \circ \eta(X) = \phi \circ \eta(Y) = Y^{\downarrow}.$$

Now assume $X^{\downarrow} = Y^{\downarrow}$. Then

$$\eta(X) = \eta \circ \phi \circ \eta(X) = \eta(X^{\downarrow}) = \eta(Y^{\downarrow}) = \eta \circ \phi \circ \eta(Y) = \eta(Y),$$

as desired. ∎

Claim 8.6.10 *For $F \in \mathcal{F}(\Delta^{NK}(\lambda))$,*

$$\{s \in S : \ \exists F' \in \mathcal{F}(\Delta^{NK}(\lambda)),\ F' \xrightarrow{s} F\} = \{s \in \phi(F) : \ \phi(F) - \{s\} \textit{ is biclosed}\}.$$

Moreover, for adjacent facets F, F', if $F' \xrightarrow{s} F$, then $\eta(\phi(F) - \{s\}) = F'$.

Proof: We first show the forward inclusion. Let $s \in S$ and $F' \in \mathcal{F}(\Delta^{NK}(\lambda))$ adjacent to F with $F' \xrightarrow{s} F$. Let $p \in F - F'$, $p' \in F' - F$, so p and p' kiss along s.

Assume $\phi(F) - \{s\}$ is not biclosed. If $\phi(F) - \{s\}$ is not closed, then $s = s_1 \circ \cdots \circ s_l$ for some $s_i \in A_{p_i}$ and $p_i \in F$ with $l > 1$. If $s_1 \notin A_p$, then p_1 and p' are kissing, an impossibility. Let $k \geq 1$ and suppose $s_1 \circ \cdots \circ s_k$ is in A_p but not $s_1 \circ \cdots \circ s_{k+1}$. Then p_{k+1} and p' are kissing, which is again impossible. Hence, $s_1 \circ \cdots \circ s_{l-1}$ is in A_p. Since p' leaves s_l to the South and enters from the West, p' and s_l are kissing, a contradiction. Hence, $\phi(F) - \{s\}$ is closed.

Now assume $\phi(F) - \{s\}$ is not co-closed. We may assume that there exists $t \in S - \phi(F)$ such that $s \circ t \in \phi(F)$ but $t \notin \phi(F)$. We prove $t \in \phi(F)$ by induction on the length of t, which gives a contradiction.

Then $s \circ t = u_1 \circ \cdots \circ u_l$ where $u_i \in A_{p_i}$ and $p_i \in F$ for all i. If p leaves u_1 to the South, then p' and p_1 are kissing, an impossibility. As before, we determine that $s \circ t$ is a SW-subsegment of p. As s is a SW-subsegment of p, this implies that the edge e between s and t is horizontal. Moreover, p is the bottom path at e in F. Let e_1 be the horizontal edge after t_{term}, and let q_1 be the bottom path at e_1. Then q_1 and p agree along a terminal subsegment t_1 of t, where p enters t_1 from the West and q_1 enters t_1 from the North. Moreover, $s \circ (t - t_1) \in \phi(F)$, so $t - t_1 \in \phi(F)$ by induction. But $t_1 \in A_{q_1}$, so $t = (t - t_1) \circ t_1 \in \phi(F)$.

Now we prove the reverse inclusion. Let $s \in \phi(F)$ such that $\phi(F) - \{s\}$ is biclosed. We prove that $\eta(\phi(F) - \{s\})$ is adjacent to F and $\eta(\phi(F) - \{s\}) \xrightarrow{s} F$.

Since $\phi(F) = \phi(\eta \circ \phi(F))$ and $\phi(F) - \{s\}$ is co-closed, there exists a path p such that $s \in A_p$. Let e be the vertical edge above s_{init}. Since $s \in A_p$, it follows that p contains e.

We show that p is the top path at e. Suppose not, and let q be the top path at e. If q does not contain s, then let t be the largest subsegment of s along which p and q agree. Then q leaves t to the East and p leaves t to the South, so $t \in A_q$ and $s - t \in A_p$. But this is impossible since $\phi(F) - \{s\}$ is co-closed.

Suppose q contains s and let v be the first vertex after s such that q leaves v to the East and p leaves v to the South. (We note that if q and p agree after s, then we may deduce a contradiction in a similar way where we take v to occur before s.) Let t be the segment $p[s_{\text{init}}, v]$. Since $t \in A_q$ and $\phi(F) - \{s\}$ is co-closed, $t - s$ is in $\phi(F)$. Let $u_1, \ldots, u_l$ be segments such that $u_i \in A_{p_i}$ for some paths p_i and $t - s = u_1 \circ \cdots \circ u_l$. Since p and p_1 are non-kissing, p must leave u_1 to the East. Similarly, we deduce that p leaves $u_2, \ldots, u_l$ to the East. But p leaves u_l to the South, a contradiction.

We have now determined that $p = q$. As p was chosen as an arbitrary path containing s as a SW-subsegment, it follows that p is the unique such path.

Let e be the edge above s_{init} and e' the edge below s_{term}. By definition, $\eta(\phi(F) - \{s\})$ differs from F by at most two paths, namely p_e and $p_{e'}$. We claim p_e is in F and it is the top path in F at e'. As $p \notin \eta(\phi(F) - \{s\})$, it would follow that F and $\eta(\phi(F) - \{s\})$ are adjacent facets and the new path in $\eta(\phi(F) - \{s\})$ kisses p along s.

Let q be the top path in F at e'. Since $\phi(F)$ contains all SW-subsegments of s and p is the top path in F at e, the path p_e contains s (by definition) and leaves s to the South. In particular p_e contains e'.

Suppose $p_e \neq q$ and let v be the first vertex after s such that p_e and q leave in different directions. (As before, if p_e and q agree after s, then we may apply a similar argument where p_e and q enter some vertex v before s in different directions.) Let t be the segment $q[(e')_{\text{term}}, v]$. If p_e leaves v to the South and q to the East, then $t \in A_q$. But this implies $s \circ t \in \phi(F)$ so p_e must leave $s \circ t$ to the East, a contradiction. On the other hand, if p_e leaves v to the East and q to the South, then $s \circ t \in \phi(F)$. As $\phi(F) = \phi(F)^{\downarrow}$, this implies $t \in \phi(F)$. In particular, there exist segments $t_1, \ldots, t_l$ such that $t = t_1 \circ \cdots \circ t_l$ and $t_i \in A_{p_i}$ for some paths p_i in F. Since q is on top at e', it must leave t_1 to the East. Since q and p_2 are non-kissing, it leaves t_2 to the East as well. Similarly, it leaves $t_3, \ldots, t_l$ to the East, a contradiction. ∎

Theorem 8.6.11 $\mathrm{GT}(\lambda)$ *is a lattice quotient of* $\mathrm{Bic}(S)$.

Proof: By Claim 8.6.9, the fibers of η are the equivalence classes of Θ. By Claim 8.6.10, the defining relation of $\mathrm{GT}(\lambda)$ coincides with the covering relations of $\mathrm{Bic}(S)/\Theta$. Therefore, $\mathrm{GT}(\lambda)$ is a well-defined partial order which is isomorphic to $\mathrm{Bic}(S)/\Theta$. ∎

As $\mathrm{Bic}(S)$ is a congruence-uniform lattice and congruence-uniformity is preserved by lattice quotients, Theorem 8.0.4 follows from Theorem 8.6.11.

8.7 Congruence structure of Grid-Tamari orders

Since $\mathrm{GT}(\lambda)$ is congruence-uniform, its collection of lattice congruences is relatively easy to compute. We carry out that computation in this section.

As in previous sections, we fix a shape λ and let S denote the its set of segments, ordered by inclusion.

Theorem 8.7.1 $\mathrm{Con}(\mathrm{GT}(\lambda))$ *and* $\mathcal{O}(S^*)$ *are isomorphic as lattices.*

Before proving Theorem 8.7.1, we establish a few simple results.

Claim 8.7.2 *For* $s \in S$, $\eta(A_s)$ *is join-irreducible.*

Proof: For $s \in S$, if $\eta(A_s) = F \vee \mathcal{F}'$, then

$$A_s = \phi \circ \eta(A_s) = \phi(F \vee \mathcal{F}') = \phi(F) \vee \phi(F').$$

Since A_s is join-irreducible in $\mathrm{Bic}(\lambda)$, we deduce that $A_s = \phi(F)$ (or $A_s = \phi(F')$), so $\eta(A_s) = F$. Hence, $\eta(A_s)$ is join-irreducible. ∎

Let $f : S \to \mathrm{JI}(\mathrm{GT}(\lambda))$ where $f(s) = \eta(A_s)$ for $s \in S$.

Claim 8.7.3 f *is a bijection.*

Proof: Clearly f is injective, since if $\eta(A_s) = \eta(A_t)$ for some $s, t \in S$, then $A_s = \phi \circ \eta(A_s) = \phi \circ \eta(A_t) = A_t$.

Let F be a join-irreducible of $\mathrm{GT}(\lambda)$. Then $\phi(F) = \bigvee_{p \in F} A_p$, so

$$F = \eta \circ \phi(F) = \eta\left(\bigvee_{p \in F} A_p\right) = \bigvee_{p \in F} \eta(A_p).$$

Since F is join-irreducible, $F = \eta(A_p)$ for some $p \in F$. If s is the largest SW-subsegment of p, then $F = \eta(A_s) = f(s)$, as desired. ∎

Since $\mathrm{GT}(\lambda)$ is a congruence-uniform lattice, the join-irreducibles of $\mathrm{GT}(\lambda)$ are in bijection with the join-irreducibles of $\mathrm{Con}(\mathrm{GT}(\lambda))$ via the map $j \mapsto \mathrm{con}(j_*, j)$ for $j \in \mathrm{JI}(\mathrm{GT}(\lambda))$. Composing with f defines a bijection between S and join-irreducibles of $\mathrm{Con}(\mathrm{GT}(\lambda))$.

Claim 8.7.4 *Let F and F' be adjacent facets of $\Delta^{NK}(\lambda)$. If $F \xrightarrow{s} F'$, then $F \vee \eta(A_s) = F'$ and $F \wedge \eta(A_s) = \eta(A_s)_*$.*

Proof: Suppose $F \xrightarrow{s} F'$. Then all proper SW-subsegments of s is in $\phi(F)$. The claim is now immediate. ∎

As a consequence of Claim 8.7.4, if $F \xrightarrow{s} F'$, then $F \equiv F' \mod \mathrm{con}(\eta(A_s)_*, \eta(A_s))$ and $\eta(A_s)_* \equiv \eta(A_s) \mod \mathrm{con}(F, F')$.

For $X \in \mathrm{Bic}(\lambda)$, define $X^{\downarrow s} = \overline{X^{\downarrow} - S_{\geq s}}$, where $S_{\geq s}$ is the set of segments containing s. It is straight-forward to check that $X^{\downarrow s}$ is biclosed. Moreover the relation $X \equiv Y \mod \Theta_s$ if $X^{\downarrow s} = Y^{\downarrow s}$ is a lattice congruence of $\mathrm{Bic}(\lambda)$ coarser than Θ. Since $\mathrm{GT}(\lambda)$ is isomorphic to $\mathrm{Bic}(\lambda)/\Theta$, this congruence decends to a lattice congruence on $\mathrm{GT}(\lambda)$. From the discussion following Claims 8.7.3 and 8.7.4, the congruence Θ_s contracts exactly those covering relations in $\mathrm{GT}(\lambda)$ labelled by a segment t containing s.

To complete the proof of Theorem 8.7.1, it remains to show that $\mathrm{con}(\eta(A_s)_*, \eta(A_s)) = \Theta_s$.

Claim 8.7.5 *Let s, t be segments such that s is an initial or terminal subsegment of t. Then $\eta(A_t)_* \equiv \eta(A_t) \mod \mathrm{con}(\eta(A_s)_*, \eta(A_s))$.*

Proof: We assume $t = s \circ s'$ for some segment s'. The case $t = s' \circ s$ is similar.

Let $X = \overline{A_s \cup A_{s'}}$. Then X consists of segments that can be decomposed as a terminal SW-subsegment of s and an initial SW-subsegment of s'. From this observation, we deduce that the sets

$$X - \{s\}, X - \{s'\}, X - \{s,t\}, X - \{s',t\}, X - \{s,t,s'\}$$

are all biclosed. Moreover, as X constains all SW-subsegments of s, s' and t, only one of these covering relations is contracted by Θ. That is, this hexagonal subposet of $\mathrm{Bic}(S)$ is mapped to a pentagonal subposet of $\mathrm{GT}(\lambda)$ under η. In particular, there are covering relations $(Y,Z),(Y',Z')$ in $\mathrm{Bic}(S)$ not contracted by Θ and labelled s,t respectively such that $Y' \equiv Z' \mod \mathrm{con}(Y,Z)$.

Then $\eta(Y) \xrightarrow{s} \eta(Z))$, $\eta(Y') \xrightarrow{t} \eta(Z')$ are covering relations of $\mathrm{GT}(\lambda)$ since (Y,Z) and (Y',Z') are not contracted by Θ. Moreover, $\eta(Y') \equiv \eta(Z') \mod \mathrm{con}(\eta(Y),\eta(Z))$. By Claim 8.7.4, we deduce that $\eta(A_t)_* \equiv \eta(A_t) \mod \mathrm{con}(\eta(A_s)_*, \eta(A_s))$, as desired. ∎

If $s \subseteq t$, then by first extending s to an initial subsegment of t and applying Claim 8.7.5 twice, we deduce that $\eta(A_t)_* \equiv \eta(A_t) \mod \mathrm{con}(\eta(A_s)_*, \eta(A_s))$. Therefore, $\mathrm{con}(\eta(A_s)_*, \eta(A_s)) = \Theta_s$ holds, and Theorem 8.7.1 is proved.

8.8 Cambrian Lattices as Grid-Tamari orders

In this section, we recall the definition of a Cambrian lattice (of type A) as a poset of triangulations of a polygon. We then identify this lattice with the Grid-Tamari order on a double ribbon shape.

Fix $n \in \mathbb{N}$ and let Q be a directed graph whose underlying graph is a path on $n-1$ vertices. Label the vertices $v_1, \dots, v_{n-1}$ in order along this path. We define a polygon P in $\mathbb{R}^2$ with vertices $w_0, \dots, w_{n+1}$ such that

- w_i has x-coordinate i for all i,
- w_0 and w_{n+1} are above the x-axis,
- w_1 and w_n are below the x-axis, and

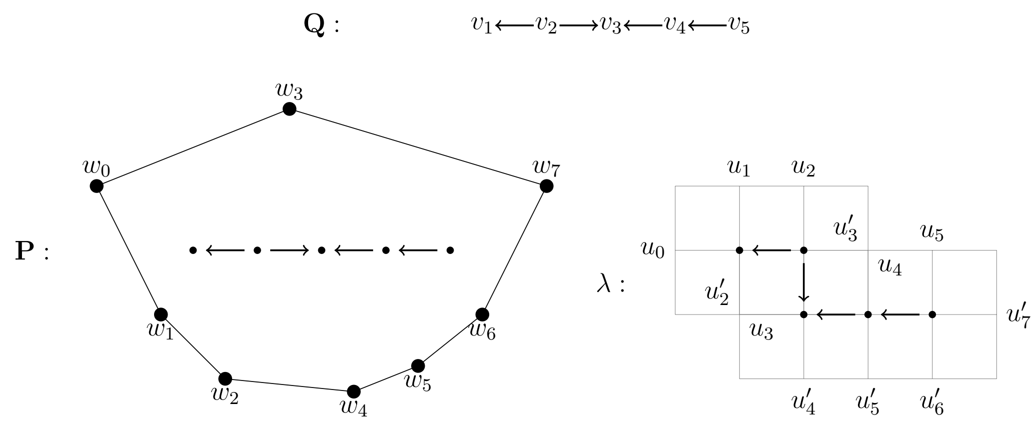


Figure 8.6: GT(λ) is a Cambrian lattice when λ is a double ribbon shape.

- for $i = 2, \ldots, n-1$, w_i is above the x-axis if and only if there is a directed edge $v_{i-1} \to v_i$ in Q.

The *Cambrian lattice* Camb(Q) is the set of triangulations of P whose covering relations are of the form $T \lessdot T'$ if T and T' differ by a single diagonal and the slope of the diagonal in $T - T'$ is less than the slope of the diagonal in $T' - T$.

Let λ be the double ribbon shape with interior vertices $v_1, \ldots, v_{n-1}$ such that for $i \in \{2, \ldots, n-1\}$,

- v_{i-1} is North of v_i if $v_{i-1} \to v_i$ in Q and
- v_{i-1} is West of v_i if $v_i \to v_{i-1}$ in Q.

Proposition 8.8.1 *Given Q and λ as above,* Camb(Q) *is isomorphic to* GT(λ) *as lattices.*

Proof: We first define a bijection between paths in λ with diagonals in P. We label the boundary vertices $u_0, \ldots, u_{n-1}$ and $u'_2, \ldots, u'_{n+1}$ where

- u_0 is West of v_1 and u_1 is North of v_1,

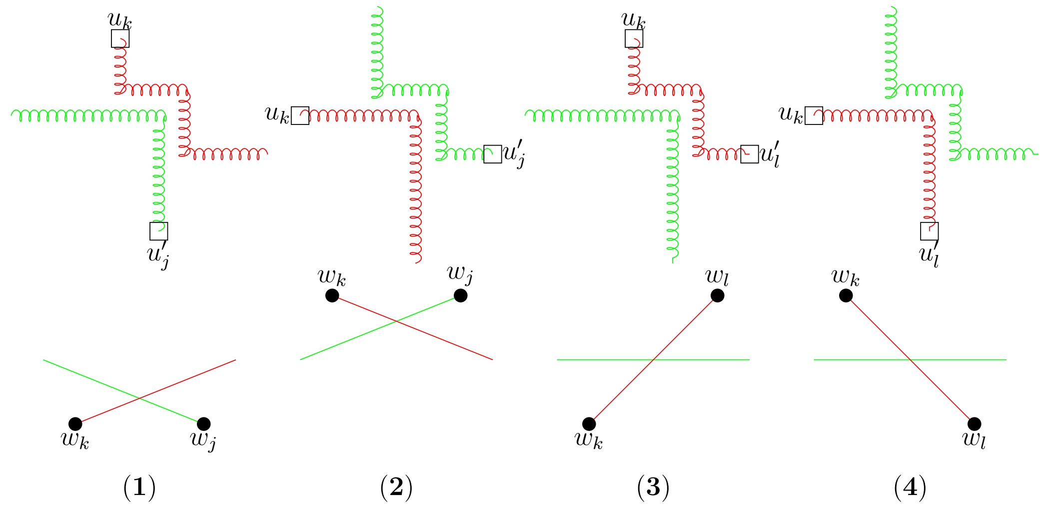


Figure 8.7: From the proof of Proposition 8.8.1: The four ways two paths may be kissing in λ and the corresponding ways two diagonals may cross in P.

- u'_{n+1} is East of v_{n-1} and u'_n is South of v_{n-1},
- for $i \in \{2, \ldots, n-1\}$, if v_{i-1} is North of v_i, then u_i is West of v_i and u'_i is East of v_{i-1}, and
- for $i \in \{2, \ldots, n-1\}$, if v_{i-1} is West of v_i, then u_i is North of v_i and u'_i is South of v_{i-1}.

Every boundary vertex that can start (end) a path is labeled u_i (u'_i) for a unique i. Let τ map paths in λ with at least one turn to diagonals of P such that the path from u_i to u'_j is sent to the diagonal between w_i and w_j. It is straight-forward to check that τ is a bijection. We check that two paths p, p' are kissing if and only if $\tau(p)$ and $\tau(p')$ are crossing.

Let p be the path between u_i and u'_j, and let p' be the path between u_k and u'_l for some i, j, k, l. Assume p and p' are kissing. Without loss of generality, we may assume $i < k$. Then exactly one of the following must hold:

1. $i < k < j < l$, u_k is North of v_k, and u'_j is South of v_{j-1};

2. $i < k < j < l$, u_k is West of v_k, and u'_j is East of v_{j-1};

3. $i < k < l < j$, u_k is North of v_k, and u'_l is East of v_{l-1}; or

4. $i < k < l < j$, u_k is West of v_k, and u'_l is South of v_{l-1}.

Similarly, the diagonal between w_i and w_j crosses the diagonal between w_k and w_l for some $i < j,\ k < l$ in exactly one of the following cases:

1. $i < k < j < l$ and w_k and w_j are below the x-axis;

2. $i < k < j < l$ and w_k and w_j are above the x-axis;

3. $i < k < l < j$, w_k is below the x-axis, and w_l is above the x-axis; or

4. $i < k < l < j$, w_k is above the x-axis, and w_l is below the x-axis.

Hence, τ induces an isomorphism of compatibility complexes. If F and F' are adjacent facets of the non-kissing complex, then there exists unique paths $p \in F - F'$ and $p' \in F - F'$. Checking the four cases above, it is routine to verify that $F < F'$ in $\mathrm{GT}(\lambda)$ if and only if the slope of $\tau(p)$ is less than the slope of $\tau(p')$. ∎

Bibliography

[1] Peter Abramenko and Kenneth S Brown. Buildings, 2008.

[2] Natalie Aisbett. Gamma-vectors of edge subdivisions of the boundary of the cross polytope. *arXiv preprint arXiv:1209.1789*, 2012.

[3] Christos A Athanasiadis. Zonotopal subdivisions of cyclic zonotopes. *Geometriae Dedicata*, 86(1-3):37–57, 2001.

[4] Christos A Athanasiadis, Paul H Edelman, and Victor Reiner. Monotone paths on polytopes. *Mathematische Zeitschrift*, 235(2):315–334, 2000.

[5] Christos A Athanasiadis, Jörg Rambau, and Francisco Santos. The generalized baues problem for cyclic polytopes ii. *Publ. de l'Institut Mathématique*, 66(80):3–15, 1999.

[6] Christos A Athanasiadis and Francisco Santos. Monotone paths on zonotopes and oriented matroids. *Canadian Journal of Mathematics*, 53(6):1121–1140, 2001.

[7] Louis J Billera, Mikhail M Kapranov, and Bernd Sturmfels. Cellular strings on polytopes. *Proceedings of the American Mathematical Society*, 122(2):549–555, 1994.

[8] Louis J. Billera and Bernd Sturmfels. Fiber polytopes. *Annals of Mathematics*, 135(3):pp. 527–549, 1992.

[9] Louis J Billera and Bernd Sturmfels. Iterated fiber polytopes. *Mathematika*, 41(2):348–363, 1994.

[10] Anders Björner. Orderings of coxeter groups. *Combinatorics and algebra*, 34:175–195, 1984.

[11] Anders Björner. Essential chains and homotopy type of posets. *Proceedings of the American Mathematical Society*, 116(4):1179–1181, 1992.

[12] Anders Björner. Topological methods, 1995.

[13] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, 2005.

[14] Anders Björner, Paul H Edelman, and Günter M Ziegler. Hyperplane arrangements with a lattice of regions. *Discrete & computational geometry*, 5(3):263–288, 1990.

[15] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M Ziegler. *Oriented matroids*, volume 46. Cambridge University Press, 1999.

[16] Anders Björner and Michelle L Wachs. Permutation statistics and linear extensions of posets. *Journal of Combinatorial Theory, Series A*, 58(1):85–114, 1991.

[17] Anders Björner and Michelle L Wachs. Shellable nonpure complexes and posets. ii. *Transactions of the American Mathematical Society*, 349:3945–3976, 1997.

[18] Jochen Bohne. *Eine kombinatorische Analyse zonotopaler Raumaufteilungen*. Sonderforschungsbereich 343, 1992.

[19] Nicolas Bourbaki. Groupes et algebras de lie, chap. 4, 5, et 6. *Hermann, Paris*, 1968.

[20] Victor M Buchstaber and Vadim D Volodin. Combinatorial 2-truncated cubes and applications. In *Associahedra, Tamari Lattices and Related Structures*, pages 161–186. Springer, 2012.

[21] Nathalie Caspard, Claude Le Conte de Poly-Barbut, and Michel Morvan. Cayley lattices of finite coxeter groups are bounded. *Advances in Applied Mathematics*, 33(1):71–94, 2004.

[22] Nathalie Caspard and C Le Conte Poly-Barbut. Tamari lattices are bounded: a new proof. Technical report, Technical Report TR-2004-03, LACL, Université Paris XII, 2004.

[23] Aldo Conca, Serkan Hosten, and Rekha R Thomas. Nice initial complexes of some classical ideals. *Contemporary Mathematics*, 423:11–42, 2007.

[24] R Cordovil and ML Moreira. A homotopy theorem on oriented matroids. *Discrete mathematics*, 111(1):131–136, 1993.

[25] Michael Cuntz and Istvan Heckenberger. Finite weyl groupoids. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2010.

[26] Alan Day. Congruence normality: the characterization of the doubling class of convex sets. *Algebra Universalis*, 31(3):397–406, 1994.

[27] Patrick Dehornoy and Marc Autord. On the distance between the expressions of a permutation. *European Journal of Combinatorics*, 31(7):1829–1846, 2010.

[28] Pierre Deligne. Les immeubles des groupes de tresses généralisés. *Inventiones mathematicae*, 17(4):273–302, 1972.

[29] Aristophanes Dimakis and Folkert Müller-Hoissen. Kp solitons, higher bruhat and tamari orders. In *Associahedra, Tamari Lattices and Related Structures*, pages 391–423. Springer, 2012.

[30] Matthew Dyer. On the weak order of coxeter groups. *arXiv preprint arXiv:1108.5557*, 2011.

[31] Matthew Dyer, Christophe Hohlweg, and Vivien Ripoll. Imaginary cones and limit roots of infinite coxeter groups. *arXiv preprint arXiv:1303.6710*, 2013.

[32] Paul H Edelman. A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes. *Transactions of the American Mathematical Society*, 283(2):617–631, 1984.

[33] Paul H Edelman and Robert E Jamison. The theory of convex geometries. *Geometriae dedicata*, 19(3):247–270, 1985.

[34] Paul H Edelman, Jörg Rambau, and Victor Reiner. On subdivision posets of cyclic polytopes. *European Journal of Combinatorics*, 21(1):85–101, 2000.

[35] Paul H Edelman and Victor Reiner. Not all free arrangements are $k(\pi, 1)$. *Bulletin of the American Mathematical Society*, 32(1):61–65, 1995.

[36] Paul H Edelman and James W Walker. The homotopy type of hyperplane posets. *Proceedings of the American Mathematical Society*, 94(2):221–225, 1985.

[37] Michael Falk and Richard Randell. The lower central series of a fiber-type arrangement. *Inventiones mathematicae*, 82(1):77–88, 1985.

[38] Stefan Felsner and Helmut Weil. A theorem on higher bruhat orders. *Discrete & Computational Geometry*, 23(1):121–127, 2000.

[39] Jon Folkman and Jim Lawrence. Oriented matroids. *Journal of Combinatorial Theory, Series B*, 25(2):199–236, 1978.

[40] Haya Friedman and Dov Tamari. Problèmes d'associativité: Une structure de treillis finis induite par une loi demi-associative. *Journal of Combinatorial Theory*, 2(3):215–242, 1967.

[41] Winfried Geyer. *Intervallverdopplung und verwandte Konstruktionen bei Verbänden*. PhD thesis, Fachbereich Mathematik der Technischen Hochschule Darmstadt, 1992.

[42] Mikhail A Gorsky. Subword complexes and edge subdivisions. *Proceedings of the Steklov Institute of Mathematics*, 286(1):114–127, 2014.

[43] George Grätzer. *The congruences of a finite lattice: a proof-by-picture approach*. Springer Science & Business Media, 2007.

[44] Branko Grünbaum. A catalogue of simplicial arrangements in the real projective plane. *Ars Mathematica Contemporanea*, 2(1), 2009.

[45] István Heckenberger and Volkmar Welker. Geometric combinatorics of weyl groupoids. *Journal of Algebraic Combinatorics*, 34(1):115–139, 2011.

[46] Andre Henriques and David E Speyer. The multidimensional cube recurrence. *Advances in mathematics*, 223(3):1107–1136, 2010.

[47] Patricia Hersh and Karola Meszaros. Sb-labelings and posets with each interval homotopy equivalent to a sphere or a ball. *arXiv preprint arXiv:1407.5311*, 2014.

[48] Christophe Hohlweg and Jean-Philippe Labbé. On inversion sets and the weak order in coxeter groups. *arXiv preprint arXiv:1502.06926*, 2015.

[49] Christophe Hohlweg, Jean-Philippe Labbé, and Vivien Ripoll. Asymptotical behaviour of roots of infinite coxeter groups i. *arXiv preprint arXiv:1112.5415*, 2011.

[50] Samuel Huang and Dov Tamari. Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law. *Journal of Combinatorial Theory, Series A*, 13(1):7–13, 1972.

[51] Danièle Huguet. La structure du treillis des polyèdres de parenthésages. *Algebra universalis*, 5(1):82–87, 1975.

[52] James E Humphreys. *Reflection groups and Coxeter groups*, volume 29. Cambridge university press, 1992.

[53] Ken Ito. The classification of convex orders on affine root systems. 2001.

[54] Michel Jambu and Luis Paris. Combinatorics of inductively factored arrangements. *European Journal of Combinatorics*, 16(3):267–292, 1995.

[55] Misha M Kapranov and Vladimir A Voevodsky. 2-categories and zamolodchikov tetrahedra equations. In *Proc. Symp. Pure Math*, volume 56, pages 177–260, 1994.

[56] Dmitry N Kozlov. Simple homotopy types of hom-complexes, neighborhood complexes, lovász complexes, and atom crosscut complexes. *Topology and its Applications*, 153(14):2445–2454, 2006.

[57] Jean-Philippe Labbé. *Polyhedral combinatorics of Coxeter groups*. PhD thesis, Berlin, Freie Universitat Berlin, Diss., 2013, 2013.

[58] Michel Las Vergnas. Extensions ponctuelles dune géométrie combinatoire orientée. In *Problemes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, pages 265–270, 1978.

[59] C Le Conte de Poly-Barbut. Sur les treillis de coxeter finis. *Mathématiques, informatique et sciences humaines*, 125:41–57, 1994.

[60] Bernard Leclerc and Andrei Zelevinsky. Quasicommuting families of quantum pliicker coordinates. In *Kirillov's seminar on representation theory*, volume 35, page 85. Amer Mathematical Society, 1998.

[61] Carl W Lee. The associahedron and triangulations of the n-gon. *European Journal of Combinatorics*, 10(6):551–560, 1989.

[62] Jean-Louis Loday. Realization of the stasheff polytope. *Archiv der Mathematik*, 83(3):267–278, 2004.

[63] Yurii I Manin and Vadim V Schechtman. Higher bruhat orders, related to the symmetric group. *Functional Analysis and Its Applications*, 20:148–150, 1986.

[64] Yurii I Manin and Vadim V Schechtman. Arrangements of hyperplanes, higher braid groups and higher bruhat orders. *Algebraic number theory*, 17:289–308, 1989.

[65] Thomas McConville. Crosscut-simplicial lattices. *arXiv preprint arXiv:1409.6269*, 2014.

[66] Thomas McConville. Gallery posets of supersolvable arrangements. *DMTCS Proceedings*, (01):815–826, 2014.

[67] Henri Mühle. Sb-labelings, distributivity, and bruhat order on sortable elements. *ArXiv e-prints*, July 2014.

[68] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300. Springer Science & Business Media, 2013.

[69] T Kyle Petersen, Pavlo Pylyavskyy, and David E Speyer. A non-crossing standard monomial theory. *Journal of Algebra*, 324(5):951–969, 2010.

[70] Annette Pilkington. Convex geometries on root systems. *Communications in Algebra®*, 34(9):3183–3202, 2006.

[71] Alex Postnikov, Victor Reiner, and Lauren Williams. Faces of generalized permutohedra. *Documenta Mathematica*, 13(207-273):51, 2008.

[72] Jörg Rambau. A suspension lemma for bounded posets. *Journal of Combinatorial Theory, Series A*, 80(2):374 – 379, 1997.

[73] Jörg Rambau and Victor Reiner. A survey of the higher stasheff-tamari orders. In *Associahedra, Tamari Lattices and Related Structures*, pages 351–390. Springer, 2012.

[74] Nathan Reading. Lattice and order properties of the poset of regions in a hyperplane arrangement. *Algebra Universalis*, 50(2):179–205, 2003.

[75] Nathan Reading. Lattice congruences, fans and hopf algebras. *Journal of Combinatorial Theory, Series A*, 110(2):237–273, 2005.

[76] Nathan Reading. Cambrian lattices. *Advances in Mathematics*, 205(2):313–353, 2006.

[77] Nathan Reading. From the tamari lattice to cambrian lattices and beyond. In *Associahedra, Tamari Lattices and Related Structures*, pages 293–322. Springer, 2012.

[78] Nathan Reading. Finite Coxeter groups and the weak order. In *Lattice Theory: Selected Topics and Applications. Volume 2.* Birkhäuser/Springer Basel AG, Basel, forthcoming.

[79] Nathan Reading. Lattice theory of the poset of regions. In *Lattice Theory: Selected Topics and Applications. Volume 2.* Birkhäuser/Springer Basel AG, Basel, forthcoming.

[80] Nathan Reading and David Speyer. Sortable elements in infinite coxeter groups. *Transactions of the American Mathematical Society*, 363(2):699–761, 2011.

[81] Vic Reiner and Yuval Roichman. Diameter of graphs of reduced words and galleries. *Trans. Amer. Math. Soc.*, 365:2779–2802, 2013.

[82] Victor Reiner. Signed posets. *Journal of Combinatorial Theory, Series A*, 62(2):324–360, 1993.

[83] Victor Reiner. The generalized baues problem. *New perspectives in algebraic combinatorics*, 38:293–336, 1999.

[84] Victor Reiner and Volkmar Welker. On the charney–davis and neggers–stanley conjectures. *Journal of Combinatorial Theory, Series A*, 109(2):247–280, 2005.

[85] Jürgen Richter-Gebert and Günter M Ziegler. Zonotopal tilings and the bohne-dress theorem. *Contemporary Mathematics*, 178:211–211, 1994.

[86] Gian-Carlo Rota. On the foundations of combinatorial theory i. theory of möbius functions. *Probability theory and related fields*, 2(4):340–368, 1964.

[87] Mario Salvetti. Topology of the complement of real hyperplanes in n. *Inventiones mathematicae*, 88(3):603–618, 1987.

[88] Luigi Santocanale and Friedrich Wehrung. Lattices of regular closed subsets of closure spaces. *arXiv preprint arXiv:1307.1480*, 2013.

[89] Francisco Santos, Christian Stump, and Volkmar Welker. Noncrossing sets and a grassmann associahedron. *arXiv preprint arXiv:1403.8133*, 2014.

[90] B Shapiro, M Shapiro, and A Vainshtein. Connected components in the intersection of two open opposite schubert cells. *International Mathematics Research Notices*, 1997(10):469–493, 1997.

[91] Richard P Stanley. Supersolvable lattices. *Algebra Universalis*, 2(1):197–217, 1972.

[92] John R Stembridge. Irreducible circuits and coxeter arrangements. *Journal of Combinatorial Theory, Series A*, 114(7):1220–1237, 2007.

[93] Bernd Sturmfels. Grobner bases and convex polytopes, university lecture series, vol. 8. *American Mathematical Society, Providence, RI*, 1996.

[94] Bernd Sturmfels and Günter M. Ziegler. Extension spaces of oriented matroids. *Discrete & Computational Geometry*, 10:23–45, 1993.

[95] Dov Tamari. Sur quelques problèmes d'associativité. *Annales scientifiques de l'Université de Clermont. Mathématiques*, 24(3):91–107, 1964.

[96] Dov Tamari, Folkert Müller-Hoissen, Jean Marcel Pallo, and James D Stasheff. *Associahedra, Tamari Lattices and Related Structures: Tamari Memorial Festschrift*. Birkhäuser, Springer, 2012.

[97] Hugh Thomas. Maps between higher bruhat orders and higher stasheff-tamari posets. *arXiv preprint math/0212097*.

[98] Jacques Tits. Le probleme des mots dans les groupes de coxeter. In *Symposia Mathematica (INDAM, Rome, 1967/68)*, volume 1, pages 175–185, 1969.

[99] James W Walker. Homotopy type and euler characteristic of partially ordered sets. *European Journal of Combinatorics*, 2(4):373–384, 1981.

[100] James W Walker. Canonical homeomorphisms of posets. *European Journal of Combinatorics*, 9(2):97–107, 1988.

[101] Takemi Yanagimoto and Masashi Okamoto. Partial orderings of permutations and monotonicity of a rank correlation statistic. *Annals of the Institute of Statistical Mathematics*, 21(1):489–506, 1969.

[102] Günter M. Ziegler. Higher bruhat orders and cyclic hyperplane arrangements. *Topology*, 32:259–279, 1993.

[103] Günter M Ziegler. Lectures on polytopes, 1995.